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$\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds with and without discrete torsion

Matthias Klein

Department of Applied Mathematics and Theoretical Physics
CMS, Wilberforce Road, Cambridge, CB3 0WA, UK
E-Mail: m.klein@damtp.cam.ac.uk

Raúl Rabadán

Departamento de Física Teórica C-XI and Instituto de Física Teórica C-XVI
Universidad Autónoma de Madrid, Cantoblanco, 28049 Madrid, Spain
E-Mail: rabadan@delta.ft.uam.es

Abstract: We discuss compact four-dimensional $\mathbb{Z}_N \times \mathbb{Z}_M$ type-IIB orientifolds. We take a systematic approach to classify the possible models and construct them explicitely. The supersymmetric orientifolds of this type have already been constructed some time ago. We find that there exist several consistent orientifolds for each of the discrete groups $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_6$, $\mathbb{Z}_2 \times \mathbb{Z}_6'$ and $\mathbb{Z}_6 \times \mathbb{Z}_6$ if anti-D5-branes are introduced. Supersymmetry is broken by the open strings ending on antibranes. The rank of the gauge group is reduced by a factor two if the underlying orbifold space has discrete torsion.

Keywords: D-branes, Superstring Vacua, Supersymmetry Breaking.

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Contents

1. Introduction

Type-IIB orientifolds [1, 2] provide a promising framework for semi-realistic model building. $D = 4$, $\mathcal{N} = 1$ type-IIB orientifolds are well studied by now [3]–[11]. In contrast to heterotic orbifolds, the action of the orbifold group on the gauge degrees of freedom of orientifolds is rather constrained and in many cases even uniquely determined due to the RR tadpole cancellation conditions. For some discrete groups that lead to $D = 4$, $\mathcal{N} = 1$ heterotic orbifolds there is no consistent supersymmetric orientifold at all. Some of the tadpole conditions vanish in the limit where the internal orbifold space gets very large. This allows for many additional consistent orientifolds. An important subset of non-compact orientifolds were classified in [12]. The action of the orientifold group $\Gamma \times \{\mathbb{1}, \Omega\}$ on the Chan-Paton matrices is specified by choosing a projective real representation γ of the orbifold group Γ [13, 12]. If Γ contains elements g of even order, i.e. the smallest positive integer N, such that $g^N = e$, is even, where e is the neutral element of Γ, then there are two inequivalent choices for the representation matrix: $\gamma_g^N = \mu\,\mathbb{1}$, with $\mu = \pm 1$. Orientifold models with $\mu = +1$ ($\mu = -1$) have been called to have (no) vector structure in [14]. If Γ contains two generators g, h of even even order, then it is possible to have non-commuting γ matrices: $\gamma_g \gamma_h = \epsilon\, \gamma_h \gamma_g$. Orientifolds with $\epsilon = -1$ have discrete torsion in the sense of [15, 16]. Thus, to each orbifold group Γ with two generators of even order, there correspond eight non-compact orientifold models, characterised by the three signs μ_g, μ_h and ϵ. If the internal space is compact and only D-branes with positive RR charge are present, then only the choice $\mu_g = \mu_h = \epsilon = -1$ is consistent [3, 6]. It is not possible to cancel the tadpoles for the remaining seven models.

A striking feature of orientifolds is the existence of a natural mechanism to break supersymmetry by introducing antibranes [17, 18]. The negative RR charge changes the GSO projection for open strings stretched between branes and antibranes, and in the antibrane-antibrane R sector it changes the Ω-projection. As a consequence, the fermionic spectrum generically differs from the bosonic spectrum in the antibrane sectors of the orientifold. This has led to orientifold models that come very close to the non-supersymmetric Standard Model [19]. The presence of two types of D-branes, with positive and negative RR charge, gives rise to many new consistent orientifolds. Indeed, the tadpoles of most of the non-compact models constructed in [12] can also be cancelled in the compact case if antibranes are introduced. The non-supersymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold models have been analysed in [20]. A supersymmetric version of these orientifolds is possible if D5-branes with negative NSNS and RR charge inside D9-branes with positive charges are introduced [11].

The aim of this paper is to classify the possible $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds with $\mathcal{N} = 1$ supersymmetry in the closed string sector and to determine their spectra explicitely. This can be viewed as a generalisation of the work of [21] to type-IIB orientifolds. For discrete groups $\mathbb{Z}_N \times \mathbb{Z}_M$, with N and M odd, there is only one compact orientifold: the supersymmetric $\mathbb{Z}_3 \times \mathbb{Z}_3$ constructed in the second ref. of [5]. For each of the two discrete groups $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_6$, there are two orientifold models, with and without vector structure. The cases without vector structure contain only positively charged D-branes. They have been constructed in [6]. The $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6'$ orientifold with vector structure has been analysed in [11]. Therefore, we restrict ourselves to even N and M in this article.

As stated above, there are eight orientifold models for each discrete group. However only six of them are inequivalent for $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_6$, and only four of them are inequivalent for $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_6'$ and $\mathbb{Z}_6 \times \mathbb{Z}_6$. The orbifold groups containing a $\mathbb{Z}_4$ factor are special in some respects. It is not possible to build the corresponding orientifolds if only positively charged D-branes are present [6]. The reason is that there exists no projective representation of $\mathbb{Z}_N \times \mathbb{Z}_4$ on the D9-branes and D5$_i$-branes, $i = 1, 2, 3$, that respects all the consistency constraints. We will see below that the the tadpole amplitudes only factorise if the charges α_i of the three sets of D5$_i$-branes satisfy $\alpha_1\alpha_2\alpha_3 = -1$. Thus, all consistent $\mathbb{Z}_N \times \mathbb{Z}_4$ orientifolds necessarily involve D5$_i$-branes with negative RR charge. Surprisingly, we find that even if antibranes are introduced, only the models with vector structure in the $\mathbb{Z}_4$ factor have a solution to the tadpole equations.[1]

There are many phase factors (more precisely, signs) that appear in type-IIB orientifolds. In section 2, we explain their significance and classify the inequivalent models. In particular, we discuss the relations between vector structure, discrete torsion, D-brane charges and O-plane charges. In section 3, we review and generalise a straightforward method to construct supersymmetric and non-supersymmetric type-IIB orientifolds. The closed string spectrum is determined using the orbifold cohomology [22, 23, 12]. To obtain the open string spectrum, we generalise the algorithm developed in [11] based on quiver theory [24] and on the formalism of [13]. We also give the general formulae for the tadpole cancellation conditions including all the possible sign factors. The algorithm described provides a useful tool for orientifold model building, especially if implemented in a computer algebra program. The complete spectra of the models discussed are presented in section 4. In two appendices, we explain how to obtain the Hodge numbers of orbifolds with and without discrete torsion using the cohomology and the closed string spectrum using the shift formalism.

2. Phase factors in orientifolds

Due to the fact that the γ matrices form a *projective* representation of the orbifold group Γ [16], there appear many phase factors in the expressions relevant to type-IIB orientifolds. These phases are normally chosen to take some convenient values.

In this section, we want to answer the question how many inequivalent choices there are. To this end, we first include all possible phase factors and then identify the factor systems that lead to equivalent models.

[1] This statement is valid if no Wilson lines are present. There might well be consistent $\mathbb{Z}_N \times \mathbb{Z}_4$ orientifolds without vector structure in the $\mathbb{Z}_4$ factor when appropriate Wilson lines are added. In this article we only consider models without Wilson lines.

2.1 Discrete torsion on orbifolds and orientifolds

We briefly review some basic results about discrete torsion.[2] In the closed string sector of orbifold theories, discrete torsion appears as phases $\beta_{g,h}$ in the one-loop partition function [15]:

$$Z = \frac{1}{|\Gamma|} \sum_{g,h\in\Gamma} \beta_{g,h}\, Z(g,h)\,, \tag{2.1}$$

where $Z(g,h)$ is the contribution of the (g,h)-twisted sector and Γ is the orbifold group. The discrete torsion phases must satisfy

$$\beta_{g,g} = 1\,, \qquad \beta_{g,h} = \beta^{-1}_{h,g}\,, \qquad \beta_{g,hk} = \beta_{g,h}\beta_{g,k}\,, \quad \forall\, g,h,k \in \Gamma, \tag{2.2}$$

to be consistent with modular invariance [15]. If we restrict ourselves to abelian orbifold groups and to complex three-dimensional internal spaces, then discrete torsion is only possible for $\Gamma = \mathbb{Z}_N \times \mathbb{Z}_M$. Let g_1, g_2 be the generators of $\mathbb{Z}_N, \mathbb{Z}_M$, respectively, $p = \gcd(N,M)$ and $\epsilon = e^{2\pi i m/p}$, with $m = 1,\ldots,p$. If we choose $\beta_{g_1,g_2} = \epsilon$, then all the other phases $\beta_{g,h}$ are fixed by (2.2). Introducing the notation $(a,b) = g_1^a g_2^b$ for the elements of Γ, they read

$$\beta_{(a,b),(c,d)} = \epsilon^{ad-bc}\,. \tag{2.3}$$

In the following, we will characterise the discrete torsion either by the parameter ϵ of (2.3) or by the number s, defined as the smallest positive integer such that $\epsilon^s = 1$.

The matrices γ_g that represent the action of the elements g of the orbifold group Γ on the Chan-Paton indices of the open strings form a projective representation:

$$\gamma_g \gamma_h = \alpha_{g,h} \gamma_{gh}\,, \tag{2.4}$$

where $\alpha_{g,h}$ are arbitrary non-zero complex numbers. They are called the factor system of the projective representation of Γ. Discrete torsion in the open string sector means that the matrices γ_g do not commute. More precisely [16, 26], this non-commutativity is controlled by the phases $\beta_{g,h}$ defined above

$$\gamma_g \gamma_h = \beta_{g,h} \gamma_h \gamma_g\,. \tag{2.5}$$

Two matrices γ_g and $\hat\gamma_g$ are considered projectively equivalent if there exists a non-zero complex number ρ_g such that $\hat\gamma_g = \rho_g \gamma_g$. In general, the projective representations γ and $\hat\gamma$ have different factor systems, but they have the same discrete torsion phases $\beta_{g,h}$. From (2.4) and (2.5), we find

$$\beta_{g,h} = \alpha_{g,h} \alpha^{-1}_{h,g}\,. \tag{2.6}$$

To perform explicit calculations, it is convenient to choose a specific factor system. We choose the $\alpha_{g,h}$ in (2.4) such that

$$(\gamma_{g_1})^a (\gamma_{g_2})^b = \gamma_{g_1^a g_2^b}\,, \quad a = 1,\ldots,N\,, \quad b = 1,\ldots,M\,, \tag{2.7}$$

[2] For a recent geometric treatment of discrete torsion, see [25].

where g_1, g_2 are the generators of $\mathbb{Z}_N \times \mathbb{Z}_M$. This choice is possible for each equivalence class of factor systems. Moreover, all the remaining phases $\alpha_{g,h}$ are then determined by the parameter ϵ. We find

$$\gamma_{g_1^a g_2^b} \gamma_{g_1^c g_2^d} = \epsilon^{-bc} \gamma_{g_1^{a+c} g_2^{b+d}} . \tag{2.8}$$

We conclude that for a $\mathbb{Z}_N$ orbifold all the possible choices of factor systems $\alpha_{g,h}$ in (2.5) are equivalent. For $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, there are $p/2+1$ (if p is even) or $(p+1)/2$ (if p is odd) non-equivalent choices, parametrised by $\epsilon = e^{2\pi i m/p}$, $m = 0, \ldots, \lceil p/2 \rceil$, where $p = \gcd(N, M)$. (Note that ϵ and ϵ^{-1} lead to equivalent orbifolds.)

In orientifold models, the matrices γ_g have to satisfy the additional constraint [7, 12]

$$\gamma_\Omega \gamma_g^* \gamma_\Omega^{-1} = c\, \delta_g\, \gamma_g \,, \tag{2.9}$$

where $\gamma_\Omega = c\, \gamma_\Omega^\top$ is the matrix that represents the action of the world-sheet parity Ω on the Chan-Paton indices and δ_g is a new phase, defined by

$$\gamma_{\Omega g}^{-1} \gamma_{\Omega g}^\top = \delta_g (\gamma_g^\top)^2 \,. \tag{2.10}$$

It is easy to see that one can always choose $\delta_g = c\, \forall\, g$. Consider the redefinition $\gamma_g \to \hat{\gamma}_g = \rho_g \gamma_g$, where the phases ρ_g satisfy $\rho_g \rho_h = \rho_{gh}$. This redefinition leaves (2.4) and (2.7) invariant and therefore does not modify the factor system $\alpha_{g,h}$. Now, choosing $\rho_g = \sqrt{c\, \delta_g}$ and using $c = \pm 1$, we find from (2.9) that $\hat{\delta}_g = c$.

The condition (2.9) together with $c\, \delta_g = 1$ tells us that the matrices γ_g form a *real* or *pseudo-real* projective representation of Γ [13, 12].

In general, there are several sets of Dp-branes, $p = 9, 5_1, 5_2, 5_3$, and correspondingly several sets of γ matrices, $\gamma_{g,p}$, $\gamma_{\Omega,p}$. It turns out (e.g. when considering tadpole cancellation) that the discrete torsion parameter ϵ cannot be chosen independently for the different sets of γ matrices. There is only one ϵ which is the same for all p. Furthermore, the coefficients c_p, defined by $\gamma_{\Omega,p}^\top = c_p\, \gamma_{\Omega,p}$, cannot be chosen at will. Tadpole cancellation requires[3] $c_9 = 1$ and a consistent coupling of the strings stretching between 9- and 5_i-branes implies [3]

$$c_{5_1} = c_{5_2} = c_{5_3} = -c_9 \,. \tag{2.11}$$

These constraints can be circumvented in models similar to the one proposed by the authors of [27, 28]. However, such models are only consistent if all matter at integer mass levels in the 95_i sectors is projected out. We will therefore discard this possibility in the following.

As a consequence of the reality condition (2.9), the discrete torsion parameter ϵ can only take the real values ± 1 in the orientifold case [12]. At this point, we might conclude that there are only two non-equivalent orientifold models corresponding to each discrete group $\mathbb{Z}_N \times \mathbb{Z}_M$. But we will see in the next subsection that there are more possibilities.

[3] if no anti-D9-branes are present

2.2 Vector structure

For $\Gamma = \mathbb{Z}_N$, the identity $g_1^N = e$, where g_1 is the generator and e the neutral element of $\mathbb{Z}_N$, translates to

$$(\gamma_{g_1})^N = \mu\,\mathbb{1}\,, \quad \text{with } \mu \in \mathbb{C}^*\,(\text{orbifold}) \quad \text{or} \quad \mu \in \mathbb{R}^*\,(\text{orientifold})\,. \tag{2.12}$$

In the orbifold case, this can be brought to the form $(\hat{\gamma}_{g_1})^N = \mathbb{1}$ by redefining the γ matrices. In the orientifold case, such a redefinition is only possible for odd N ($\gamma_{g_1} \to \hat{\gamma}_{g_1} = \mathrm{sgn}(\mu)|\mu|^{-1/N}\gamma_{g_1}$). If N is even, the two cases $\mu = +1$ and $\mu = -1$ are not equivalent because $(\gamma_{g_1})^N = -\mathbb{1}$ cannot be brought to the form $(\hat{\gamma}_{g_1})^N = \mathbb{1}$ by multiplying γ_{g_1} with a real number. For the factor system of (2.7), this means that $\gamma_{g_1^N} = \mu\,\gamma_e$. If $\mu = -1$, then the matrices γ_g represent the elements of Γ only 2:1. This class of orientifolds has been called to have no vector structure in [14].

For $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds, the notion of vector structure can be defined for each of the two factors of the discrete group separately.[4] There are four inequivalent models, characterised by $\mu_{3,p} = \pm 1$, $\mu_{1,p} \pm 1$, where

$$(\gamma_{g_1,p})^N = \mu_{3,p}\,\mathbb{1}\,, \qquad (\gamma_{g_2,p})^M = \mu_{1,p}\,\mathbb{1}\,. \tag{2.13}$$

The indices $i = 1, 3$ on $\mu_{i,p}$ refer to the fact that, in our conventions, g_1 fixes the third complex plane and g_2 fixes the first complex plane. The index $p = 9, 5_1, 5_2, 5_3$ indicates which set of Dp-branes is considered. We will see below that the signs $\mu_{i,p}$ cannot be chosen independently for each set of Dp-branes. Indeed, the choice of $\mu_{1,9}$ and $\mu_{3,9}$ fixes all other $\mu_{i,p}$.

We conclude that to each $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifold there correspond eight distinct models characterised by the three signs $\mu_{1,9}$, $\mu_{3,9}$ and ϵ.

2.3 Phases in the open string one-loop amplitudes

The three open string one-loop amplitudes that contribute to the tadpoles — the cylinder, the Klein bottle and the Möbius strip — contain additional signs. However, the requirement of tadpole cancellation relates these signs to $\mu_{1,9}$, $\mu_{3,9}$ and ϵ.

In the following, we label the elements of $\mathbb{Z}_N \times \mathbb{Z}_M$ by the two-vector $\bar{k} = (a, b)$, $a = 0, \ldots, N-1$, $b = 0, \ldots, M-1$. We define $s_i = \sin(\pi\bar{k}\cdot\bar{v}_i)$, $c_i = \cos(\pi\bar{k}\cdot\bar{v}_i)$ and $\tilde{s}_i = \sin(2\pi\bar{k}\cdot\bar{v}_i)$, as in section 3.3 below.

The RR part of the cylinder amplitude can be written in the form [12, 11]:

$$\mathcal{C} = \sum_{\bar{k}=(0,0)}^{(N-1,M-1)} \mathcal{C}_{(\bar{k})} = \sum_{\bar{k}=(0,0)}^{(N-1,M-1)} \frac{1}{8 s_1 s_2 s_3}\left[\mathrm{Tr}\,\gamma_{\bar{k},9} + 4\sum_{i=1}^{3}\alpha_i\, s_j s_k\, \mathrm{Tr}\,\gamma_{\bar{k},5_i}\right]^2\,, \tag{2.14}$$

[4] Strictly speaking, a $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifold only has vector structure in the sense of [14, 29] if $(\gamma_{g_1})^N = (\gamma_{g_2})^M = \mathbb{1}$ and $\epsilon = 1$. We use the word "vector structure" in a generalised sense.

where we suppressed the volume dependence.[5] The indices j, k take values such that (ijk) is a permutation of (123). The signs α_i weight the 95_i sectors relative to the 99 and 5_i5_i sectors. They are related to the D-brane charges. If the D9-branes have positive RR charge, then α_i is the RR charge of the D5_i-branes.

The RR part of the Klein bottle amplitude can be written in the form:

$$\mathcal{K} = \sum_{\bar{k}=(0,0)}^{(N-1,M-1)} \mathcal{K}_{(\bar{k})} = \sum_{\bar{k}=(0,0)}^{(N-1,M-1)} \left[16 \prod_{i=1}^{3} \frac{2c_i^2}{\tilde{s}_i} - 16 \sum_{i=1}^{3} \epsilon_i \, \frac{2c_i^2}{\tilde{s}_i} \right] . \tag{2.15}$$

The sign ϵ_i weights the i-th order-two twisted sector relative to the untwisted sector.[6] It is related to the O-plane charges. If the O9-planes have negative RR charge, then ϵ_i is (half) the RR charge of the O5_i-planes.

The RR part of the Möbius strip amplitude can be written in the form:

$$\begin{aligned} \mathcal{M} = \sum_{\bar{k}=(0,0)}^{(N-1,M-1)} \mathcal{M}_{(\bar{k})} = \sum_{\bar{k}=(0,0)}^{(N-1,M-1)} &\left[-8 \prod_{i=1}^{3} \frac{1}{2s_i} \operatorname{Tr}\left(\gamma_{\Omega\bar{k},9}^{-1} \gamma_{\Omega\bar{k},9}^{\top}\right) - \right. \\ &\left. - 8 \sum_{i=1}^{3} \frac{2c_j c_k}{s_i} \, (-\alpha_i) \operatorname{Tr}\left(\gamma_{\Omega\bar{k},5_i}^{-1} \gamma_{\Omega\bar{k},5_i}^{\top}\right) \right] . \end{aligned} \tag{2.16}$$

From the interpretation of these open string one-loop amplitudes as closed string tree-level exchange between D-branes and O-planes, it is clear that the tadpoles must factorise, i.e. $\mathcal{C}+\mathcal{K}+\mathcal{M}$ can be written as a sum of squares. More precisely, we need

$$\mathcal{C}_{(2\bar{k})} + \mathcal{K}_{(\bar{k})} + \sum_{i=1}^{3} \mathcal{K}_{(\bar{k}+\bar{k}_i)} + \mathcal{M}_{(\bar{k})} + \sum_{i=1}^{3} \mathcal{M}_{(\bar{k}+\bar{k}_i)} = [\cdots]^2 \, , \tag{2.17}$$

where $\bar{k}_i$ denotes the order-two twist that fixes the i-th complex plane. This implies a relation between the ϵ_i and the vector structures and discrete torsion, defined above. We find [12]

$$\epsilon_i = \mu_{i,9} = \mu_{i,5_i} = -\mu_{i,5_j} \, , \quad i = 1, 3, \; j \neq i \, , \qquad \epsilon_2 = \epsilon_1 \epsilon_3 \, \epsilon^{-MN/4} \, . \tag{2.18}$$

A further constraint on the signs arises from the requirement of tadpole cancellation in the untwisted sector:

$$\epsilon_i = -\alpha_i \, , \quad i = 1, 2, 3 \, . \tag{2.19}$$

This means that the tadpoles can only be cancelled if the RR charge of the O5_i-planes (ϵ_i) is opposite to the RR charge of the D5_i-branes (α_i).

We conclude that the signs α_i, ϵ_i are not independent parameters. As stated above, there are eight distinct $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds. They can be characterised by $\mu_{1,9}$, $\mu_{3,9}$ and ϵ. Note that the three signs ϵ_i are only independent if N and M are not multiples of four.

[5] In sectors with fixed planes, this formula has to be modified as explained in [11, appendix B].

[6] These signs were introduced in the twisted Klein bottle amplitude by the authors of [20]. Note that the ϵ_i of [20] are $-\epsilon_i$ in our notation.

JHEP10(2000)049

3. Construction of the models

We consider compact orientifolds of the form $\mathbb{T}^6/(\Gamma \times \{\mathbb{1}, \Omega\})$, with $\Gamma = \mathbb{Z}_N \times \mathbb{Z}_M$, N, M even. The six-torus is defined as $\mathbb{T}^6 = \mathbb{C}^3/\Lambda$, with Λ a factorisable lattice, i.e. it is the direct sum of three two-dimensional lattices. The world-sheet symmetry Ω is of the form

$$\Omega = \Omega_0 J T \,, \tag{3.1}$$

where Ω_0 is the world-sheet parity and the operator J exchanges the k-th and the $(N-k)$-th twisted sector [30]. The operator T acts as $-\epsilon_i \mathbf{1}$ on the g-twisted states if g is an order-two twist that fixes the i-th complex plane and as the identity on the remaining states.[7] The signs ϵ_i are defined in (2.15).

The action of Γ on the coordinates (z_1, z_2, z_3) of $\mathbb{C}^3$ can be characterised by the twist vectors $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3)$:

$$g_1: \; z_i \longrightarrow e^{2\pi i v_i} z_i \,, \qquad g_2: \; z_i \longrightarrow e^{2\pi i w_i} z_i \,, \tag{3.2}$$

where g_1, g_2 are the generators of Γ and $\sum_{i=1}^3 v_i = \sum_{i=1}^3 w_i = 0$ to ensure $\mathcal{N} = 1$ supersymmetry of the closed string sector in $D = 4$. Not all possible shifts correspond to a symmetry of some lattice. Indeed, there is a finite number of $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds [21]. For even N and M, there are only 6 models: $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_6$, $\mathbb{Z}_6 \times \mathbb{Z}_6$, with twist vectors $v = 1/N(1,-1,0)$, $w = 1/M(0,1,-1)$, and $\mathbb{Z}_2 \times \mathbb{Z}_6'$ with twist vectors $v = 1/N(1,-1,0)$, $w = 1/M(-2,1,1)$. We chose the twist vectors such that $g_1^{N/2}$ fixes the third complex plane and $g_2^{M/2}$ fixes the first complex plane.

Supersymmetric compact $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds have been discussed in [3, 5, 6, 8]. The author of [6] found that the models with discrete groups $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$ are not consistent because the algebra of $\gamma_{\Omega,p}$ and $\gamma_{g,p}$, where $p = 9, 5_i$, seems to imply that $\gamma_{R,9}$ is antisymmetric, where R is an order-two twist. It turns out that there is no projective representation of $\mathbb{Z}_N \times \mathbb{Z}_M$, with $\gamma_{R,9}$ antisymmetric, if N and/or M is a multiple of four [6]. However, the condition that $\gamma_{R,9}$ be antisymmetric can be escaped if the action of the operator T (i.e. the signs ϵ_i) in (3.1) is modified. We will see that there are solutions to the tadpole equations for all orbifold groups $\mathbb{Z}_N \times \mathbb{Z}_M$ if one chooses the appropriate signs ϵ_i.

In this section, we sketch the basic steps to construct $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds. We explain how to obtain the closed string spectrum, the open string spectrum and the tadpole cancellation conditions. This is very similar to, and in fact a straightforward generalisation of, the method presented in [11].

[7] The analogous operator in $D = 6$ was discussed by the authors of [27, 30] when analysing a new $\mathbb{Z}_2$ orientifold.

Γ	ϵ		untw.	order-two	fixed pl. but not order-two	no fixed pl.	total
$\mathbb{Z}_2 \times \mathbb{Z}_2$	1	$h^{1,1}$	3	48	-	-	51
		$h^{2,1}$	3	0	-	-	3
	-1	$h^{1,1}$	3	0	-	-	3
		$h^{2,1}$	3	48	-	-	51
$\mathbb{Z}_2 \times \mathbb{Z}_4$	1	$h^{1,1}$	3	34	8	16	61
		$h^{2,1}$	1	0	0	0	1
	-1	$h^{1,1}$	3	18	0	0	21
		$h^{2,1}$	1	0	8	0	9
$\mathbb{Z}_2 \times \mathbb{Z}_6$	1	$h^{1,1}$	3	22	10	16	51
		$h^{2,1}$	1	0	2	0	3
	-1	$h^{1,1}$	3	0	8	8	19
		$h^{2,1}$	1	14	4	0	19
$\mathbb{Z}_2 \times \mathbb{Z}'_6$	1	$h^{1,1}$	3	18	0	15	36
		$h^{2,1}$	0	0	0	0	0
	-1	$h^{1,1}$	3	0	0	12	15
		$h^{2,1}$	0	15	0	0	15
$\mathbb{Z}_3 \times \mathbb{Z}_3$	1	$h^{1,1}$	3	-	54	27	84
		$h^{2,1}$	0	-	0	0	0
	$e^{\pm 2\pi i/3}$	$h^{1,1}$	3	-	0	0	3
		$h^{2,1}$	0	-	27	0	27
$\mathbb{Z}_3 \times \mathbb{Z}_6$	1	$h^{1,1}$	3	4	36	30	73
		$h^{2,1}$	0	1	0	0	1
	$e^{\pm 2\pi i/3}$	$h^{1,1}$	3	4	0	6	13
		$h^{2,1}$	0	1	12	0	13

Table 1: Contribution from the different sectors to the Hodge numbers of $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds with and without discrete torsion.

3.1 Closed string spectrum

The closed string spectrum can be obtained from the cohomology of the internal orbifold space [22, 23, 12, 11]. In appendix A, we explain in detail how to obtain these numbers and the explicit contribution from each twisted sector. The results are shown in tables 1 and 2. For completeness, we also give the Hodge numbers of $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_6$. An alternative method to obtain the closed string spectrum is presented in appendix B.

Let us remark that the Hodge numbers of tables 1 and 2 directly give the closed string spectrum of the orbifold. It is obtained by dimensionally reducing to $D = 4$ the massless spectrum of type-IIB supergravity in $D = 10$: the metric g, the NSNS

Γ	ϵ		untw.	order-two	fixed pl. but not order-two	no fixed pl.	total
$\mathbb{Z}_4 \times \mathbb{Z}_4$	1	$h^{1,1}$	3	27	24	36	90
		$h^{2,1}$	0	0	0	0	0
	-1	$h^{1,1}$	3	27	0	12	42
		$h^{2,1}$	0	0	0	0	0
	$\pm i$	$h^{1,1}$	3	3	0	0	6
		$h^{2,1}$	0	0	12	0	12
$\mathbb{Z}_6 \times \mathbb{Z}_6$	1	$h^{1,1}$	3	12	30	39	84
		$h^{2,1}$	0	0	0	0	0
	-1	$h^{1,1}$	3	0	24	24	51
		$h^{2,1}$	0	3	0	0	3
	$e^{\pm 2\pi i/3}$	$h^{1,1}$	3	12	0	12	27
		$h^{2,1}$	0	0	3	0	3
	$e^{\pm 2\pi i/6}$	$h^{1,1}$	3	0	0	6	9
		$h^{2,1}$	0	3	6	0	9

Table 2: Contribution from the different sectors to the Hodge numbers of $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds with and without discrete torsion. (continued)

2-form B, the dilaton ϕ, the RR-forms $C^{(0)}$, $C^{(2)}$, $C^{(4)}$. (We only give the bosons, the fermions are related to them by supersymmetry.) The Lorentz indices of the 10D fields are contracted with the differential forms of the internal space. The resulting spectrum has $\mathcal{N} = 2$ supersymmetry in $D = 4$. For a general configuration one finds (see e.g. [12, 31]):

- a gravity multiplet (consisting of the graviton, a vector and fermions)
- a double tensor multiplet (consisting of two 2-forms, two scalars and fermions)
- $h^{1,1}$ tensor multiplets (consisting of a 2-form, three scalars and fermions)
- $h^{2,1}$ vector multiplets (consisting of a vector, two scalars and fermions)

To obtain the orientifold spectrum, one has to perform the Ω-projection. In the following, we will denote an element $g = g_1^a g_2^b$ of $\mathbb{Z}_N \times \mathbb{Z}_M$ by the two-vector $\bar{k} = (a, b)$, i.e. $(0, 0)$ is the neutral element, $(1, 0)$ is the generator of $\mathbb{Z}_N$, $(0, 1)$ the generator of $\mathbb{Z}_M$, etc. As explained in [12, 11], it is convenient to split the sectors into three different types:

(i) The untwisted sector ($\bar{k} = (0, 0)$), it is invariant under J and T. The bosonic fields in $D = 4$ are found contracting the Lorentz indices of the Ω-even 10D fields $g_{\mu\nu}$, ϕ, $C^{(2)}_{\mu\nu}$ with the harmonic forms corresponding to $h^{0,0}$, $h^{3,0}$, $h^{1,1}_{\rm untw}$, $h^{2,1}_{\rm untw}$.

(ii) The order-two sectors ($\bar{k}_1 = (0, M/2)$, $\bar{k}_2 = (N/2, M/2)$, $\bar{k}_3 = (N/2, 0)$), they are invariant under J but acquire an extra sign $-\epsilon_i$ under the action of T. In the sector $\bar{k}_i$, one has to distinguish the two cases $\epsilon_i = \pm 1$. If $\epsilon_i = -1$, then the Ω-even 10D fields $g_{\mu\nu}$, ϕ, $C^{(2)}_{\mu\nu}$ are contracted with $h^{1,1}_{\bar{k}_i}$, $h^{2,1}_{\bar{k}_i}$. If $\epsilon_i = +1$, then the Ω-odd 10D fields $B_{\mu\nu}$, $C^{(4)}_{\mu\nu\rho\sigma}$ are contracted with $h^{1,1}_{\bar{k}_i}$, $h^{2,1}_{\bar{k}_i}$.

(iii) The remaining sectors. To get the fields in $D = 4$, one forms linear combinations of the harmonic forms that belong to the k-th and $(N-k)$-th twisted sector. The J-even forms are contracted with the Ω-even 10D fields and the J-odd forms are contracted with the Ω-odd 10D fields.

The spectrum fits into $\mathcal{N} = 1$ supermultiplets. In total, one finds:

(i) the gravity multiplet, a linear multiplet, $(h^{1,1} + h^{2,1})_{\mathrm{untw}}$ chiral multiplets.

(ii) for each $i = 1, 2, 3$: $h^{1,1}_{\bar{k}_i} + h^{2,1}_{\bar{k}_i}$ chiral multiplets if $\epsilon_i = -1$ and $h^{1,1}_{\bar{k}_i}$ linear multiplets and $h^{2,1}_{\bar{k}_i}$ vector multiplets if $\epsilon_i = +1$.

(iii) if the $\bar{k}$-twisted sector has fixed planes, then there are $\frac{1}{2} h^{1,1}_{\bar{k}}$ linear multiplets, $\frac{1}{2} h^{2,1}_{\bar{k}}$ vector multiplets and $\frac{1}{2}(h^{1,1}_{\bar{k}} + h^{2,1}_{\bar{k}})$ chiral multiplets, else one has $h^{1,1}_{\bar{k}}$ linear multiplets.

As explained in [12], orbifolds with non-real discrete torsion have twisted sectors $\bar{k}$, where $h^{2,1}_{\bar{k}} \neq h^{1,2}_{\bar{k}}$. The world-sheet symmetry Ω, eq. (3.1), is not a symmetry of such orbifolds. Therefore, consistent orientifolds can only be constructed from orbifolds with discrete torsion $\epsilon = \pm 1$.

3.2 Open string spectrum

There are 32 D9-branes and 32 D5_i-branes for each $i = 1, 2, 3$. The index i indicates that the 5_i-branes fill the four non-compact directions and the i-th complex plane.

The action of Γ on the Chan-Paton indices of the open strings is described by a projective representation $\gamma^{(p)}$ that associates a (32×32)-matrix $\gamma_{g,p}$ to each element g of Γ, where $p = 9, 5_i$ denotes the type of D-brane the open strings end on:

$$\begin{aligned} \gamma^{(p)} :\ \Gamma &\longrightarrow \mathrm{GL}(32, \mathbb{C})\,, \\ g &\longmapsto \gamma_{g,p}\,. \end{aligned} \tag{3.3}$$

Because of the orientifold projection, this representation must be real or pseudo-real. In general, $\gamma^{(p)}$ can be decomposed in irreducible blocks of real (R^r), pseudo-real (R^p) and complex (R^c) representations [13]:

$$\gamma^{(p)} = \left(\bigoplus_{l_1} n^r_{l_1} R^r_{l_1} \right) \oplus \left(\bigoplus_{l_2} n^p_{l_2} R^p_{l_2} \right) \oplus \left(\bigoplus_{l_3} n^c_{l_3} (R^c_{l_3} \oplus \bar{R}^c_{l_3}) \right). \tag{3.4}$$

In this expression, the notation $n_l R_l$ is short for $R_l \otimes \mathbb{1}_{n_l}$, i.e. n_l is the number of copies of the irreducible representation (irrep) R_l in $\gamma^{(p)}$ [13].

JHEP10(2000)049

Let us first consider the 99 and 5_i5_i sectors. The projection on invariant states of the Chan-Paton matrices $\lambda^{(0)}$ that correspond to gauge bosons (NS sector) and their fermionic partners (R sector) in $D = 4$ imposes the constraints (see e.g. [7, 18])

$$\lambda^{(0)} = \gamma_{g,p}\lambda^{(0)}\gamma_{g,p}^{-1}\,, \quad \forall\, g \in \Gamma\,, \qquad \lambda^{(0)} = -\alpha_p\, \gamma_{\Omega,p}\lambda^{(0)\,\top}\gamma_{\Omega,p}^{-1}\,, \tag{3.5}$$

where α_p is the charge of the Dp-branes. More precisely, if we consider the fermionic (bosonic) components of $\lambda^{(0)}$ in (3.5), then α_p is the RR (NSNS) charge. The sign α_p in (3.5) can be explained as follows. In [32], it was found that the action of Ω acquires an additional minus sign in the R sector if the RR charge of the Dp-branes is reversed. This change in Ω manifests itself in the Möbius strip amplitude which is proportional to the RR charge of the Dp-branes. It is easy to see that a sign flip in the NSNS charge leads to an additional minus sign for the action of Ω in the NS sector.

Let us make some further comments on $\alpha_p^{\rm NS}$. Supersymmetry breaking is a consequence of the fact that the RR charge of the antibranes differs from their NSNS charge. In [11], it was shown that a supersymmetric version of the orientifolds discussed in [20] is possible if instead of antibranes one introduces a new type of D-branes with negative NSNS and RR charge. These objects — called D5$^-$-branes in [11] — live inside D9-branes with positive NSNS and RR charge. It is not yet clear, whether D5$^-$-branes exist in string theory. However, we would like to stress that once the possibility of a sign flip in the NSNS charge is accepted everything fits nicely together. Orientifolds involving D5$^-$-branes have the same fermionic spectrum as their non-supersymmetric analogues based on antibranes. The sign flip in the NSNS charge leads to modified projection conditions for the bosons which results in a supersymmetric spectrum.

The constraints (3.5) are easily solved. One finds that the gauge group on the Dp-branes is [13]

$$G_{(p)} = \prod_{l_1} \mathrm{SO}(n^r_{l_1}) \times \prod_{l_2} \mathrm{USp}(n^p_{l_2}) \times \prod_{l_3} U(n^c_{l_3}) \tag{3.6}$$

if $\gamma_{\Omega,p} = \alpha_p^{\rm NS}\,\gamma_{\Omega,p}^{\top}$ and

$$G_{(p)} = \prod_{l_1} \mathrm{USp}(n^r_{l_1}) \times \prod_{l_2} \mathrm{SO}(n^p_{l_2}) \times \prod_{l_3} U(n^c_{l_3}) \tag{3.7}$$

if $\gamma_{\Omega,p} = -\alpha_p^{\rm NS}\,\gamma_{\Omega,p}^{\top}$. The fermions transform in the same representation as the gauge bosons if the gauge group is unitary or $\alpha_p^{\rm R} = \alpha_p^{\rm NS}$, else they transform in the symmetric (antisymmetric) representation if the the gauge bosons transform in the antisymmetric (symmetric) representation.

In our case, $\Gamma = \mathbb{Z}_N \times \mathbb{Z}_M$. There are NM/s^2 irreps, all of them s-dimensional, where s is the smallest positive integer such that $\epsilon^s = 1$ [33]. In the case without discrete torsion, $\epsilon = s = 1$, they are of the form

$$\begin{aligned} &R_{k,l}(g_1) = e^{\pi i(2k+\eta_1)/N}\,, \qquad R_{k,l}(g_2) = e^{\pi i(2l+\eta_2)/M}\,, \\ &\text{with} \quad k = 0, \ldots, N-1\,, \qquad l = 0, \ldots, M-1\,, \end{aligned} \tag{3.8}$$

where $\eta_{1/2} = 0$ or 1, depending on whether the first/second factor of the discrete group has vector structure or not. One has $\eta_1 = (1 - \mu_{3,p})/2$, $\eta_2 = (1 - \mu_{1,p})/2$, where the $\mu_{i,p}$ are defined in (2.13). In the case with discrete torsion, $\epsilon = -1$, $s = 2$, the irreps are of the form (see e.g. [33])

$$\begin{aligned} &R_{k,l}(g_1) = e^{\pi i(2k+\eta_1)/N} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad R_{k,l}(g_2) = e^{\pi i(2l+\eta_2)/M} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ &\text{with} \quad k = 0, \ldots, N/2-1\,,\ l = 0, \ldots, M/2-1\,. \end{aligned} \tag{3.9}$$

The action of Γ on the internal $\mathbb{C}^3$ is described by a representation R_{C^3}:

$$\begin{aligned} \gamma:\ R_{C^3} &\longrightarrow \mathrm{SU}(3)\,, \\ g &\longmapsto R_{C^3}(g)\,. \end{aligned} \tag{3.10}$$

We write $R_{C^3} = R_{C^3}^{(1)} \oplus R_{C^3}^{(2)} \oplus R_{C^3}^{(3)}$ (this decomposition is possible whenever Γ is abelian), where $R_{C^3}^{(i)}$ corresponds to the action of R_{C^3} on the i-th coordinate of $\mathbb{C}^3$. In the notation of eq. (3.2), we have

$$R_{C^3}^{(i)}(g_1) = e^{2\pi i v_i}\,, \qquad R_{C^3}^{(i)}(g_2) = e^{2\pi i w_i}\,. \tag{3.11}$$

The matter fields corresponding to the i-th complex plane are obtained from the projections (see e.g. [7, 18])

$$\lambda^{(i)} = R_{C^3}^{(i)}(g)\gamma_{g,p}\lambda^{(i)}\gamma_{g,p}^{-1}\,, \quad \forall\, g \in \Gamma\,, \quad \lambda^{(i)} = \alpha_p\, R_\Omega^{(i)}\gamma_{\Omega,p}\lambda^{(i)\top}\gamma_{\Omega,p}^{-1}\,, \tag{3.12}$$
$$\text{with } R_\Omega^{(i)} = \begin{cases} -1 & \text{if } p = 9 \text{ or } p = 5_i \\ +1 & \text{if } p = 5_j \text{ and } j \neq i \end{cases} \quad \text{and} \quad \alpha_p = \text{charge of D}p\text{-brane}\,.$$

In general, the fermionic spectrum (R sector) may differ from the bosonic spectrum (NS sector). In the R sector (NS sector) α_p is the RR (NSNS) charge of the Dp-branes. These equations can be solved using quiver theory [24], as explained in [11, appendix A].

Let us now consider the 95_i and 5_i5_j sectors. The projection of the open strings $\lambda^{(95_i)}$, stretching from 9-branes to 5_i-branes, on Γ-invariant states reads (see e.g. [7, 18]):

$$\begin{aligned} &\text{R sector: } \lambda^{(95_i)} = \left(R_{C^3}^{(i)}(g)\right)^{-\alpha_i^{\mathrm{R}}/2} \gamma_{g,9}\lambda^{(95_i)}\gamma_{g,5_i}^{-1}\,, \\ &\text{NS sector: } \lambda^{(95_i)} = \left(R_{C^3}^{(j)}(g)\right)^{\alpha_i^{\mathrm{R}}/2} \left(R_{C^3}^{(k)}(g)\right)^{\alpha_i^{\mathrm{NS}}/2} \gamma_{g,9}\lambda^{(95_i)}\gamma_{g,5_i}^{-1}\,, \end{aligned} \tag{3.13}$$

where $\alpha_i^{\rm R}$ ($\alpha_i^{\rm NS}$) is the RR (NSNS) charge of the D5_i-branes and (ijk) is a cyclic permutation of (123).[8] In the supersymmetric case $\alpha_i^{\rm R} = \alpha_i^{\rm NS}$, the conditions on the fermionic and bosonic spectrum coincide as they should.[9] Ω relates the 95_i sector with the 5_i9 sector and does not impose extra conditions on $\lambda^{(95_i)}$. Similarly, the projection on the open strings $\lambda^{(5_i5_j)}$, stretching from 5_i-branes to 5_j-branes, on Γ-invariant states reads

$$\begin{aligned}
&\text{R sector: } \lambda^{(5_i5_j)} = \left(R_{C^3}^{(k)}(g)\right)^{-\alpha_i^{\rm R}\alpha_j^{\rm R}/2} \gamma_{g,5_i}\lambda^{(5_i5_j)}\gamma_{g,5_j}^{-1}\,,\\
&\text{NS sector: } \lambda^{(5_i5_j)} = \left(R_{C^3}^{(i)}(g)\right)^{\alpha_i^{\rm R}\alpha_j^{\rm R}/2}\left(R_{C^3}^{(j)}(g)\right)^{\alpha_i^{\rm NS}\alpha_j^{\rm NS}/2} \gamma_{g,5_i}\lambda^{(5_i5_j)}\gamma_{g,5_j}^{-1}\,, \qquad (3.14)
\end{aligned}$$

The spectrum is easiest obtained using quiver theory, as explained in [11, appendix A].

3.3 Tadpoles

In this subsection we give the tadpole cancellation conditions including all the possible signs that may appear in the different contributions.[10] It is straightforward to obtain these conditions from the eqs. (2.14)–(2.16) using the methods described in [12, 11, 34].

The elements of $\mathbb{Z}_N \times \mathbb{Z}_M$ are labelled by the two-vector $\bar{k} = (a, b)$, $a = 0, \ldots, N-1$, $b = 0, \ldots, M-1$. We define $s_i = \sin(\pi\bar{k}\cdot\bar{v}_i)$, $c_i = \cos(\pi\bar{k}\cdot\bar{v}_i)$ and $\tilde{s}_i = \sin(2\pi\bar{k}\cdot\bar{v}_i)$, where $\bar{v} = (v, w)$ combines the two twist vectors of (3.2). Let us comment on the various signs appearing in the tadpole contributions (see also section 2).

- In the cylinder amplitude, α_i weights the 95_i sector relative to the 99 and 5_i5_i sectors. It gives the RR charge of the the D5_i-branes.

- In the Klein bottle amplitude, ϵ_i is related to the choice between the standard and alternative Ω-projection, eq. (3.1). It gives (half) the RR charge of the O5_i-planes.

- The signs $\mu_{i,p}$ and c_p are related to the symmetry properties of the γ matrices. They are defined by $(\gamma_{(1,0),p})^N = \mu_{3,p}\,\mathbb{1}$, $(\gamma_{(0,1),p})^M = \mu_{1,p}\,\mathbb{1}$ and $\gamma_{\Omega,p}^\top = c_p\,\gamma_{\Omega,p}$.

- The discrete torsion parameter ϵ can only take the values ±1. It is defined in section 2.1.

[8] In contrast to [18], we choose all fermions to have positive chiralities, the fermions with negative chiralities being their antiparticles.

[9] Note that $R_{C^3}^{(i)}(g)R_{C^3}^{(j)}(g)R_{C^3}^{(k)}(g) = 1$ is a consequence of $R_{C^3}^{(1)}(g)\oplus R_{C^3}^{(2)}(g)\oplus R_{C^3}^{(3)}(g) \in {\rm SU}(3)$ in (3.10).

[10] We restrict ourselves to the RR tadpoles. The formulae are valid if D9-branes and D5_i-branes or anti-D5_i-branes are present. The NSNS tadpoles have exactly the same form if the signs α_i, ϵ_i are now interpreted as the NSNS charges of the D5_i-branes, O5_i-planes, respectively.

- The sign β_i depends on the discrete torsion (2.3) via $\beta_i = \beta_{\bar{k}_i,\bar{k}}$, where $\bar{k}_i$ is the order-two twist that fixes the i-th complex plane.

Factorisation of the twisted tadpoles requires [12]

$$\epsilon_i = \mu_{i,9} = \mu_{i,5_i} = -\mu_{i,5_j}\,, \quad i=1,3,\ j\neq i\,, \qquad \epsilon_1\epsilon_2\epsilon_3 = \epsilon^{-MN/4}\,. \tag{3.15}$$

The untwisted tadpoles can only be cancelled if

$$\alpha_i = -\epsilon_i\,, \quad i=1,2,3\,. \tag{3.16}$$

This is the statement that the D-brane charges must be opposite to the O-plane charges in order to cancel the RR tadpoles. If the RR flux can escape to infinity (non-compact cases), this condition can be absent.

Furthermore, it is easy to see [2] that the action of Ω^2 on the oscillator part of strings stretching from 9-branes to 5_i-branes is related to c_9 and c_{5_i} by

$$\Omega^2|_{95_i} = c_9 c_{5_i} = -1\,. \tag{3.17}$$

The last identity follows from an argument given in [2]. In special orientifold models with no massless matter in the 95_i sectors, it is possible to have $\Omega^2|_{95_i} = +1$, as shown in [27, 28]. But in this article, we will stick to the standard Ω-action. Moreover, in order to avoid the introduction of anti-D9-branes, we choose $c_9 = 1$.

We are still free to choose the signs $\mu_{1,9}$, $\mu_{3,9}$ and ϵ. All the other signs are then fixed by the above conditions:

$$\begin{aligned}\Omega^2|_{95_j} &= -c_9 = c_{5_j} = -1\,, \quad j=1,2,3\,,\\ \mu_{i,9} &= \mu_{i,5_i} = -\mu_{i,5_j} = -\alpha_i = \epsilon_i\,, \quad i=1,3, j\neq i\,,\\ -\alpha_2 &= \epsilon_2 = \epsilon_1\epsilon_3\,\epsilon^{-MN/4}\,.\end{aligned} \tag{3.18}$$

The tadpole cancellation conditions are:

(a) untwisted sector

$$\mathrm{Tr}\,\gamma_{(0,0),9} = 32\,, \qquad \alpha_i\ \mathrm{Tr}\,\gamma_{(0,0),5_i} = 32\,c_{5_i}\,\epsilon_i\,, \quad i=1,2,3\,. \tag{3.19}$$

(b) twisted sectors without fixed tori, i.e. $\bar{k}\cdot\bar{v}_i \neq 0 \bmod \mathbb{Z}$:

- odd $\bar{k}$:

$$\mathrm{Tr}\,\gamma_{\bar{k},9} + \sum_{i=1}^{3} 4\,\alpha_i\,s_j s_k\,\mathrm{Tr}\,\gamma_{\bar{k},5_i} = 0\,, \tag{3.20}$$

where (ijk) is a permutation of (123).

- even $\bar{k} = 2\bar{k}'$:

$$\mathrm{Tr}\,\gamma_{2\bar{k}',9} + \sum_{i=1}^{3} 4\,\alpha_i\tilde{s}_j\tilde{s}_k\,\mathrm{Tr}\,\gamma_{2\bar{k}',5_i} = 32\,\epsilon^{-k'_1k'_2}\left(c_1c_2c_3 - \sum_{i=1}^{3}\epsilon_i\beta_i\,c_i s_j s_k\right), \tag{3.21}$$

where s_i, c_i, $\tilde{s}_i$ and β_i are evaluated with the argument $\bar{k}' = (k'_1, k'_2)$.

(c) twisted sectors with fixed tori, i.e. $\bar{k} \cdot \bar{v}_i = 0 \bmod \mathbb{Z}$:

- odd $\bar{k}$:

$$\mathrm{Tr}\,\gamma_{\bar{k},9} + 4\,\alpha_i\, s_j s_k\, \mathrm{Tr}\,\gamma_{\bar{k},5_i} = 0\,, \qquad \alpha_j\, \mathrm{Tr}\,\gamma_{\bar{k},5_j} = \pm\alpha_k\, \mathrm{Tr}\,\gamma_{\bar{k},5_k}\,. \tag{3.22}$$

  The sign in the second line depends on whether $\bar{k} \cdot \bar{v}_i$ is even (upper sign) or odd (lower sign).

- even $\bar{k} = 2\bar{k}'$, with $\bar{k}' \cdot \bar{v}_i = 0$:

$$\begin{aligned} \mathrm{Tr}\,\gamma_{2\bar{k}',9} + 4\,\alpha_i\, \tilde{s}_j \tilde{s}_k\, \mathrm{Tr}\,\gamma_{2\bar{k}',5_i} &= 32\,\epsilon^{-k'_1 k'_2}\,(c_j c_k - \epsilon_i \beta_i\, s_j s_k)\,, \\ \alpha_j\, \mathrm{Tr}\,\gamma_{2\bar{k}',5_j} - \alpha_k\, \mathrm{Tr}\,\gamma_{2\bar{k}',5_k} &= -8\,\epsilon^{-k'_1 k'_2}\,(\epsilon_j \beta_j - \epsilon_k \beta_k)\,. \end{aligned} \tag{3.23}$$

- even $\bar{k} = 2\bar{k}'$, with $\bar{k}' \cdot \bar{v}_i = \pm 1/2$:

$$\begin{aligned} \mathrm{Tr}\,\gamma_{2\bar{k}',9} + 4\,\alpha_i\, \tilde{s}_j \tilde{s}_k\, \mathrm{Tr}\,\gamma_{2\bar{k}',5_i} &= \mp 32\,\epsilon^{-k'_1 k'_2}\,(\epsilon_j \beta_j\, c_j^2 + \epsilon_k \beta_k\, c_k^2)\,, \\ \alpha_j\, \mathrm{Tr}\,\gamma_{2\bar{k}',5_j} + \alpha_k\, \mathrm{Tr}\,\gamma_{2\bar{k}',5_k} &= \pm 8\,\epsilon^{-k'_1 k'_2}\,(1 - \epsilon_i \beta_i)\,. \end{aligned} \tag{3.24}$$

From (3.18), we see that the orientifolds corresponding to the discrete groups $\mathbb{Z}_N \times \mathbb{Z}_M$ fall into two classes: (i) neither N nor M is a multiple of four, (ii) N and/or M is a multiple of four. If we want to introduce only standard D5$_i$-branes with positive RR charge (i.e. $\alpha_i = 1$, $i = 1, 2, 3$), then the unique solution to the type (i) orientifolds is characterised by $\mu_{1,9} = \mu_{3,9} = -1$ and $\epsilon = -1$. These are the $\mathcal{N} = 1$ supersymmetric models discussed in [3, 6, 8]. They have discrete torsion in the sense of [16, 12] which is evident from the fact that the rank of their gauge groups is reduced by a factor two. There are no models of type (ii) with only positively charged D5$_i$-branes [6]. However, many more consistent $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds are possible if one allows for antibranes. In these orientifolds, supersymmetry is broken by the open strings ending on antibranes.

The D5$_i$-branes may be distributed over different points in the j-th and k-th internal torus. Of course the tadpole equations depend on the location of the 5-branes and not all configurations are consistent. The Klein bottle contribution to the tadpoles of the $\bar{k}$-twisted sector consists of an untwisted part $\mathcal{K}_0(\bar{k})$ and three twisted parts $\mathcal{K}_i(\bar{k})$ corresponding to the three order-two twists $\bar{k}_1 = (0, M/2)$, $\bar{k}_2 = (N/2, M/2)$, $\bar{k}_3 = (N/2, 0)$. The $\mathcal{K}_0$ contribution gives the term proportional to $c_1 c_2 c_3$ in (3.21), the $\mathcal{K}_i$ contributions give the terms proportional to $\epsilon_i \beta_i\, c_i s_j s_k$. (Note that these terms are combined with the cylinder contribution to the $2\bar{k}$-twisted sector.) At each point of the internal space, the contribution $\mathcal{K}_0(\bar{k})$ is only present if this point is fixed under $\bar{k}$ and the contribution $\mathcal{K}_i(\bar{k})$ is only present if this point is fixed under $\bar{k} + \bar{k}_i$. Thus, the above tadpole cancellation conditions are strictly valid only at the points that are fixed under the whole $\mathbb{Z}_N \times \mathbb{Z}_M$. At points which are only fixed under some subgroup of $\mathbb{Z}_N \times \mathbb{Z}_M$, these conditions have to be modified accordingly.

JHEP10(2000)049

One has to analyse the tadpoles at all the fixed points in each twisted sector. If the twisted sector under consideration has fixed tori extended in the i-th direction, one has to analyse the tadpoles at each fixed point in the remaining two directions (i.e. the points where the fixed tori are located). At each fixed point all the D9-branes contribute but only those D5$_i$-branes that have this point inside their world-volume. The Klein bottle contribution to this fixed point is determined as explained in the preceding paragraph. We will see how this works in the examples below. In the $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$ models it is not possible to put all the D5$_i$-branes at the origin.

4. Description of the models

All the models contain 32 D9-branes and three sets of 32 D5$_i$-branes wrapping the i-th internal torus, $i = 1, 2, 3$. In general there are eight different models for each orbifold group $\mathbb{Z}_N \times \mathbb{Z}_M$, with N and M even. They can be characterised by the three signs $\mu_{1,9}$, $\mu_{3,9}$ (vector structures) and ϵ (discrete torsion). If neither N nor M is a multiple of four (type (i) models), one can alternatively choose the signs α_1, α_2, α_3 (RR charges of D5$_i$-branes), whereas only two of the three α_i's are independent if N and/or M is a multiple of four (type (ii) models), eq. (3.18). If $N = M$, only four of the eight possible models are inequivalent, corresponding to the number of negative α_i's. As explained in the paragraph below eq. (3.24), the orientifolds involving only standard D9-branes and D5$_i$-branes, i.e. $(\alpha_1, \alpha_2, \alpha_3) = (+1, +1, +1)$, are quite restricted. Only the type (i) models are possible. All of them are supersymmetric and have discrete torsion. They have been constructed by the authors of [3, 6, 8]. More possibilities arise if one introduces antibranes [17, 18]. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifolds with supersymmetry broken on antibranes have been discussed in [20]. In this section, we construct the remaining $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds for all possible values of $(\alpha_1, \alpha_2, \alpha_3)$. Interestingly enough, some of the models are inconsistent (in the absence of Wilson lines) because it is impossible to cancel the twisted tadpoles at all fixed points. The inconsistent models are $\mathbb{Z}_2 \times \mathbb{Z}_4$ with $(\alpha_1, \alpha_2, \alpha_3) = (+1, -1, +1)$ and $(+1, +1, -1)$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$ for all $(\alpha_1, \alpha_2, \alpha_3) \neq (-1, -1, -1)$. This inconsistency does not depend on the discrete torsion. As we will see it is related to the vector structure in each $\mathbb{Z}_4$ generator. Most probably, there exists a solution to the tadpole conditions for the above models if appropriate Wilson lines are added. But we did not consider models with Wilson lines in this article.

If D5$^-$-branes with negative NSNS and RR charge exist, then replacing the anti-D5-branes by D5$^-$-branes in the non-supersymmetric models discussed below leads to a supersymmetric version for each of these models. The fermionic spectrum of the supersymmetric orientifolds coincides with the fermionic spectrum of the orientifolds containing antibranes. The complete spectrum is obtained by replacing in the antibrane sectors USp gauge group factors by SO and changing the bosons such that they form $\mathcal{N} = 1$ multiplets with the corresponding fermions.

4.1 $\mathbb{Z}_2 \times \mathbb{Z}_2$, $v = \frac{1}{2}(1,-1,0)$, $w = \frac{1}{2}(0,1,-1)$

These orientifolds have been constructed in [3, 20]. The eight possible models can be characterised by the RR charges $(\alpha_1, \alpha_2, \alpha_3)$ of the D5$_i$-branes. Of course, two models $(\alpha_i, \alpha_j, \alpha_k)$ and $(\alpha_{i'}, \alpha_{j'}, \alpha_{k'})$ are equivalent if $(i'j'k')$ is a permutation (ijk). There are only four inequivalent $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifolds [20], corresponding to $n = 0, 1, 2, 3$ negative α_i's. From (3.18), we find that $\alpha_1\alpha_2\alpha_3 = -\epsilon$. Thus, the models with even (odd) n have (no) discrete torsion.

Using the method described in the previous section, it is very easy to construct the two $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifolds with discrete torsion. From (3.9), we find that there is a unique projective irreducible representation, which is pseudo-real if $(\mu_3, \mu_1) = (-1,-1)$ and real else. From (3.18), (3.6) and (3.7), one then finds that the $(+1,+1,+1)$ model has gauge group USp(n_0) on the D9-branes and gauge group USp(n_i) on the D5$_i$-branes, whereas the $(+1,-1,-1)$ model has gauge group SO(n_0) on the D9-branes, gauge group SO(n_1) on the D5$_1$-branes and gauge group USp(n_j) on the D5$_j$-branes, $j = 2, 3$. The tadpole conditions (3.22) for the three twisted sectors $(0,1)$, $(1,1)$, $(1,0)$ are trivially satisfied because the γ matrices are of the form (3.9), i.e. they are traceless. The untwisted tadpole conditions (3.19) imply that the rank of all gauge groups is 8, i.e. $n_0 = n_1 = n_2 = n_3 = 16$. We consider the situation where all the D5$_i$-branes are located at the origin in the internal space.

The quiver diagrams for these two models are trivial. As there is a unique projective irrep, they consist only of one node. For each of the sectors 99, 5_i5_i, $i = 1, 2, 3$, one finds 3 matter fields[11] transforming as second rank tensors under the gauge group. To decide whether these are symmetric or antisymmetric tensors, one needs to evaluate the index defined in eq. (A.9) of [11]. For each of the six mixed sectors 95_i, 5_i5_j, one finds one matter field in the bifundamental representation.

The complete spectrum of these two models is displayed in tables 3, 4.

The two $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifolds without discrete torsion — $(-1,-1,-1)$ and $(+1,+1,-1)$ — are slightly more complicated. In this case, there are four projective irreps, eq. (3.8). If $(\mu_3, \mu_1) = (+1,+1)$, then all of them are real, else they are all complex. From (3.18), (3.6) and (3.7), one then finds that the $(-1,-1,-1)$ model has a gauge group of the form SO $\times$ SO $\times$ SO $\times$ SO on the D9-branes and a gauge group of the form U $\times$ U on the D5$_i$-branes, whereas the $(+1,+1,-1)$ model has a gauge group of the form U $\times$ U on the D9-branes and on the D5$_j$-branes, $j = 1, 2$, and a gauge group of the form USp $\times$ USp $\times$ USp $\times$ USp on the D5$_3$-branes. The untwisted tadpole conditions (3.19) fix the rank of the gauge group to be 16 for each of the four sets of D-branes. Now, the twisted tadpole conditions (3.22) impose additional constraints because, generically, the γ matrices are not traceless. For the

[11]In the supersymmetric sectors, these are $\mathcal{N} = 1$ chiral multiplets. In the non-supersymmetric sectors, these are pairs of fermions and bosons which may transform in different representations.

$\mathbb{Z}_2 \times \mathbb{Z}_2,\ (\alpha_1, \alpha_2, \alpha_3) = (+1, +1, +1),\ \epsilon = -1$	
open string spectrum	
sector	gauge group / matter fields
99, $5_i 5_i$	USp(16) $3\,\genfrac{}{}{0pt}{}{\square}{\square}$
95_i, $5_i 5_j$	$(\square,\square)$
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 6 chir.
order-two	48 chir.
remaining	-

Table 3: Spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold with discrete torsion and $(\mu_3, \mu_1) = (-1, -1)$. The signs α_i are the charges of the $\mathrm{D}5_i$-branes. The open string states are in $\mathcal{N} = 1$ vector (gauge) and chiral (matter) multiplets. The spectrum in the 99 sector and the three $5_i 5_i$ sectors is identical. Thus, in total, one has four copies of the USp(16) gauge group. In the same way, there are six copies of bifundamentals connecting the four gauge group factors.

three order-two sectors $\bar{k}_1 = (0, 1)$, $\bar{k}_2 = (1, 1)$, $\bar{k}_3 = (1, 0)$, these conditions are

$$\mathrm{Tr}(\gamma_{\bar{k}_i,9}) = \mathrm{Tr}(\gamma_{\bar{k}_i,5_j}) = 0\,, \quad i, j = 1, 2, 3\,. \tag{4.1}$$

Here, we used the fact that eq. (3.22) must be imposed at each of the 16 $\bar{k}_i$ fixed points (more precisely, fixed tori extended in the i-th complex plane). As all $\mathrm{D}5_i$-branes are located at the origin, their contribution to (3.22) vanishes at some fixed points. In the $\bar{k}_i$ sector, the $\mathrm{D}5_i$-branes only contribute to the fixed point at the origin in the j-th and k-th complex plane, where (ijk) is a permutation of (123). The $\mathrm{D}5_j$-branes contribute to the four fixed points at the origin in the k-th complex plane. Therefore, the tadpole conditions take the form given in (4.1). The complete spectrum of these two models is displayed in tables 5, 6.

4.2 $\mathbb{Z}_2 \times \mathbb{Z}_4$, $v = \frac{1}{2}(1, -1, 0)$, $w = \frac{1}{4}(0, 1, -1)$

This orientifold is not symmetric under a permutation of the three sets of $\mathrm{D}5_i$-branes. The three order-two twists $\bar{k}_i$ that fix the world-volume of the $\mathrm{D}5_i$-branes are $\bar{k}_1 = (0, 2)$, $\bar{k}_2 = (1, 2)$, $\bar{k}_3 = (1, 0)$. Only $\bar{k}_1$ is of even order. As a consequence, the tadpole conditions for the $\mathrm{D}5_1$-branes differ from those for the $\mathrm{D}5_2$- and $\mathrm{D}5_3$-branes. In contrast to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold, we expect that six of the eight possible $\mathbb{Z}_2 \times \mathbb{Z}_4$ models are inequivalent. However, only for the models with vector structure in the $\mathbb{Z}_4$ factor, i.e. $\mu_1 = +1$, does a consistent solution to the tadpole equations exist, at least in the absence of Wilson lines.

JHEP10(2000)049

$\mathbb{Z}_2 \times \mathbb{Z}_2,\ (\alpha_1, \alpha_2, \alpha_3) = (+1, -1, -1),\ \epsilon = -1$	
open string spectrum	
sector	gauge group / matter fields
$99,\ 5_1 5_1$ (susy)	SO(16) $2\,\begin{smallmatrix}\square\\\square\end{smallmatrix},\ \square\square$
$5_2 5_2,\ 5_3 5_3$ (non-susy)	USp(16) spinors: $3\,\begin{smallmatrix}\square\\\square\end{smallmatrix},\ \square\square$, scalars: $2\,\square\square,\ \begin{smallmatrix}\square\\\square\end{smallmatrix}$
$95_i,\ 5_i 5_j$	$(\square,\square)$
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 6 chir.
order-two	16 chir., 32 vec.
remaining	-

Table 4: Spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold with discrete torsion and $(\mu_3, \mu_1) = (+1, -1)$. The signs α_i are the charges of the D5_i-branes. In the susy open string sectors, the gauge fields are in $\mathcal{N} = 1$ vector multiplets and the matter fields in $\mathcal{N} = 1$ chiral multiplets. In the non-susy sectors, we display the spin 0 (complex scalars) and spin 1/2 (Weyl or Majorana fermions) fields separately. In addition, there are spin 1 gauge bosons. The 99 and $5_1 5_1$ spectrum coincides and the $5_2 5_2$ and $5_3 5_3$ spectrum coincides as well. Thus, in total, one has two copies of the gauge groups SO(16) and USp(16). In the same way, there are six copies of bifundamentals connecting the four gauge group factors.

From (3.18), we find that $\alpha_1 \alpha_2 \alpha_3 = -1$. Thus, only an odd number of negative α_i's is allowed. We found a solution for $(-1, -1, -1)$ and $(-1, +1, +1)$ both with and without discrete torsion.

Let us first analyse the cases with discrete torsion. Most of the twisted tadpole conditions are trivially satisfied because the matrices $\gamma_{(1,0),p}$ and $\gamma_{(0,1),p}$ are traceless, eq. (3.9). The only non-trivial tadpole conditions are (3.19) and (3.23), corresponding to the untwisted sector and the $(0, 2)$ sector. The former fixes the rank of the gauge group to be 8 for each of the four sectors 99, $5_i 5_i$. If $\alpha_1 = -1$, then the latter reads

$$\mathrm{Tr}(\gamma_{(0,2),9}) = 0\,, \qquad \mathrm{Tr}(\gamma_{(0,2),5_1,n}) = 8\,, \qquad \mathrm{Tr}(\gamma_{(0,2),5_2}) = \mathrm{Tr}(\gamma_{(0,2),5_3}) = 0\,, \qquad (4.2)$$

where $n = 0, 1, 2, 3$ denotes the $\mathbb{Z}_4$ fixed points in the second and third complex plane. We assumed that there are no D5_1-branes at the 12 remaining $\mathbb{Z}_2$ fixed points and we located all D5_2- and D5_3-branes at the origin. By a similar reasoning to the one used at the end of section 4.1, the tadpole equations (3.23) simplify to the form given in (4.2). One can verify that the $D5_2$- and D5_3-branes have $\mu_1 = -1$ if $\alpha_1 = -1$, which implies, using (3.9), that the matrices $\gamma_{(0,2),5_{2/3}}$ are traceless. Therefore the last two conditions in (4.2) are trivially satisfied. The second condition in (4.2) forces us to put some D5_1-branes at each of the four $\mathbb{Z}_4$ fixed points. In the following, we

$\mathbb{Z}_2 \times \mathbb{Z}_2,\ (\alpha_1, \alpha_2, \alpha_3) = (-1, -1, -1),\ \epsilon = 1$	
open string spectrum	
sector	gauge group / matter fields
99 (susy)	$\mathrm{SO}(8)_1 \times \mathrm{SO}(8)_2 \times \mathrm{SO}(8)_3 \times \mathrm{SO}(8)_4$ $(\Box_1,\Box_2),\ (\Box_1,\Box_3),\ (\Box_1,\Box_4),\ (\Box_2,\Box_3),\ (\Box_2,\Box_4),\ (\Box_3,\Box_4)$
5_i5_i $i = 1, 2, 3$ (non-susy)	$\mathrm{U}(8)_1 \times \mathrm{U}(8)_2$ $\mathcal{N} = 1$ mult.: $(\Box_1,\Box_2),\ (\Box_1,\overline{\Box}_2),\ (\overline{\Box}_1,\Box_2),\ (\overline{\Box}_1,\overline{\Box}_2)$ spinors: $adj_1,\ adj_2,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_1,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_1,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_2,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_2$ scalars: $\Box\!\Box_1,\ \overline{\Box\!\Box}_1,\ \Box\!\Box_2,\ \overline{\Box\!\Box}_2$
95_1 (non-susy)	spinors: $(\Box_1,\overline{\Box}_1),\ (\Box_2,\Box_2),\ (\Box_3,\Box_1),\ (\Box_4,\overline{\Box}_2)$ scalars: $(\Box_1,\Box_2),\ (\Box_2,\overline{\Box}_1),\ (\Box_3,\overline{\Box}_2),\ (\Box_4,\Box_1)$
95_2 (non-susy)	spinors: $(\Box_1,\Box_2),\ (\Box_2,\Box_1),\ (\Box_3,\overline{\Box}_1),\ (\Box_4,\overline{\Box}_2)$ scalars: $(\Box_1,\overline{\Box}_1),\ (\Box_2,\overline{\Box}_2),\ (\Box_3,\Box_2),\ (\Box_4,\Box_1)$
95_3 (non-susy)	spinors: $(\Box_1,\Box_1),\ (\Box_2,\overline{\Box}_1),\ (\Box_3,\Box_2),\ (\Box_4,\overline{\Box}_2)$ scalars: $(\Box_1,\overline{\Box}_2),\ (\Box_2,\Box_2),\ (\Box_3,\overline{\Box}_1),\ (\Box_4,\Box_1)$
5_i5_j (susy)	$(\overline{\Box}_1,\Box_1),\ (\Box_1,\overline{\Box}_2),\ (\overline{\Box}_2,\overline{\Box}_1),\ (\Box_2,\Box_2)$
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 6 chir.
order-two	48 lin.
remaining	-

Table 5: Spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold without discrete torsion and $(\mu_3, \mu_1) = (+1, +1)$. The signs α_i are the charges of the D5_i-branes. In the susy open string sectors, the gauge fields are in $\mathcal{N} = 1$ vector multiplets and the matter fields in $\mathcal{N} = 1$ chiral multiplets. In the non-susy sectors, we display the spin 0 (complex scalars) and spin 1/2 (Weyl or Majorana fermions) fields separately. In addition, there are spin 1 gauge bosons. The spectrum in the three 5_i5_i and in the three 5_i5_j sectors is identical (our convention is that (ijk) is a cyclic permutation of (123)). Thus, in total, one has three copies of the $\mathrm{U}(8)_1 \times \mathrm{U}(8)_2$ gauge group.

will consider the most symmetric situation where eight D5_1-branes sit at each of the $\mathbb{Z}_4$ fixed points. It is straightforward to obtain the spectrum of this orientifold using the methods described in section 3.

The two cases without discrete torsion are slightly more complicated. But it is easy to see that again some D5_1-branes are needed at the four $\mathbb{Z}_4$ fixed points. The simplest consistent brane configuration is the one described in the previous paragraph. The tadpole conditions of the $(0, 2)$ sector are again given by (4.2) and the last two of these conditions are again trivially satisfied. The untwisted tadpole conditions now imply that the rank of the gauge group is 16 for each of the four sectors 99, 5_i5_i. The complete spectrum was computed with the help of a computer algebra program. The results are displayed in tables 7–10.

$\mathbb{Z}_2 \times \mathbb{Z}_2,\ (\alpha_1, \alpha_2, \alpha_3) = (+1, +1, -1),\ \epsilon = 1$	
open string spectrum	
sector	gauge group / matter fields
$99,\ 5_1 5_1,\ 5_2 5_2$ (susy)	$\mathrm{U}(8)_1 \times \mathrm{U}(8)_2$ $(\Box_1, \Box_2),\ (\Box_1, \overline{\Box}_2),\ (\overline{\Box}_1, \Box_2),\ (\overline{\Box}_1, \overline{\Box}_2),$ $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_1,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_1,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_2,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_2$
$5_3 5_3$ (non-susy)	$\mathrm{USp}(8)_1 \times \mathrm{USp}(8)_2 \times \mathrm{USp}(8)_3 \times \mathrm{USp}(8)_4$ $\mathcal{N} = 1$ mult.: $(\Box_1, \Box_2)$, $(\Box_1, \Box_3)$, $(\Box_1, \Box_4)$, $(\Box_2, \Box_3)$, $(\Box_2, \Box_4)$, $(\Box_3, \Box_4)$ spinors: $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_1,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_2,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_3,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_4$
95_1 (susy)	$(\overline{\Box}_1, \Box_1),\ (\Box_1, \Box_2),\ (\Box_2, \overline{\Box}_1),\ (\overline{\Box}_2, \overline{\Box}_2)$
95_2 (susy)	$(\Box_1, \overline{\Box}_1),\ (\overline{\Box}_1, \overline{\Box}_2),\ (\Box_2, \Box_1),\ (\overline{\Box}_2, \Box_2)$
95_3 (non-susy)	spinors: $(\Box_1, \Box_1),\ (\overline{\Box}_1, \Box_2),\ (\Box_2, \Box_3),\ (\overline{\Box}_2, \Box_4)$ scalars: $(\overline{\Box}_1, \Box_3),\ (\Box_1, \Box_4),\ (\overline{\Box}_2, \Box_1),\ (\Box_2, \Box_2)$
$5_2 5_3$ (non-susy)	spinors: $(\Box_1, \Box_1),\ (\overline{\Box}_1, \Box_3),\ (\Box_2, \Box_2),\ (\overline{\Box}_2, \Box_4)$ scalars: $(\overline{\Box}_1, \Box_2),\ (\Box_1, \Box_4),\ (\overline{\Box}_2, \Box_1),\ (\Box_2, \Box_3)$
$5_3 5_1$ (non-susy)	spinors: $(\Box_1, \overline{\Box}_2),\ (\Box_2, \overline{\Box}_1),\ (\Box_3, \Box_1),\ (\Box_4, \Box_2)$ scalars: $(\Box_1, \overline{\Box}_1),\ (\Box_2, \overline{\Box}_2),\ (\Box_3, \Box_2),\ (\Box_4, \Box_1)$
$5_1 5_2$ (susy)	$(\Box_1, \overline{\Box}_1),\ (\overline{\Box}_1, \Box_2),\ (\overline{\Box}_2, \Box_1),\ (\Box_2, \overline{\Box}_2)$
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 6 chir.
order-two	16 lin., 32 chir.
remaining	-

Table 6: Spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold without discrete torsion and $(\mu_3, \mu_1) = (+1, -1)$. The notation is explained in the previous table.

Surprisingly, we found no consistent solution for the $(+1, +1, -1)$ model with or without discrete torsion and without Wilson lines.[12] This is due to the impossibility of cancelling the tadpoles in the $(0, 2)$ sector. To write down the tadpole conditions, let us label the four $\mathbb{Z}_2$ fixed points in the second and in the third complex plane by n_2 and n_3. From (3.23) we find

$$\begin{aligned} \mathrm{Tr}(\gamma_{(0,2),9}) = \mathrm{Tr}(\gamma_{(0,2),5_1}) &= 0\,, \\ \mathrm{Tr}(\gamma_{(0,2),5_2,n_3}) + \mathrm{Tr}(\gamma_{(0,2),5_3,n_2}) &= 16\,\epsilon\,, \quad \text{if } (n_2, n_3) \text{ is fixed under } \mathbb{Z}_4\,, \\ \mathrm{Tr}(\gamma_{(0,2),5_2,n_3}) + \mathrm{Tr}(\gamma_{(0,2),5_3,n_2}) &= 0\,, \quad \text{if } (n_2, n_3) \text{ is not fixed under } \mathbb{Z}_4\,. \end{aligned} \tag{4.3}$$

The two equations in the first line of (4.3) are trivially satisfied, because both D9- and D5_1-branes have no vector structure in the $\mathbb{Z}_4$ factor. However, the last two

[12]The $(+1, +1, -1)$ and $(+1, -1, +1)$ models are equivalent up to a permutation of D5_2- and D5_3-branes.

$\mathbb{Z}_2 \times \mathbb{Z}_4,\ (\alpha_1, \alpha_2, \alpha_3) = (-1, -1, -1),\ \epsilon = -1$	
open string spectrum	
sector	gauge group / matter fields
99 (susy)	$SO(8)_1 \times SO(8)_2$ $2\,(\square_1, \square_2),\ \begin{smallmatrix}\square\\\square\end{smallmatrix}_1,\ \square\!\square_2$
$5_{1,n}5_{1,n},\ n = 0, 1, 2, 3$ (non-susy)	$USp(4)$ spinors: $2\,\begin{smallmatrix}\square\\\square\end{smallmatrix}$, scalars: $\square\!\square$
$5_25_2,\ 5_35_3$ (non-susy)	$U(8)$ spinors: $2\,adj,\ \begin{smallmatrix}\square\\\square\end{smallmatrix},\ 2\,\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}, \square\!\square$ scalars: $adj,\ \square\!\square,\ 2\,\overline{\square\!\square},\ \begin{smallmatrix}\square\\\square\end{smallmatrix}$
$95_{1,n}$ (non-susy)	spinors: $(\square_1, \square)$, scalars: $(\square_2, \square)$
95_2 (non-susy)	spinors: $(\square_1, \overline{\square}),\ (\square_2, \square)$, scalars: $(\square_1, \overline{\square}),\ (\square_2, \square)$
95_3 (non-susy)	spinors: $(\square_1, \overline{\square}),\ (\square_2, \square)$, scalars: $(\square_1, \square),\ (\square_2, \overline{\square})$
5_25_3 (susy)	$(\square, \square),\ (\overline{\square}, \overline{\square})$
$5_{1,n}5_3,\ 5_{1,n}5_2$ (susy)	$(\square, \overline{\square})$
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 4 chir.
order-two	18 lin.
remaining	4 chir., 4 vec.

Table 7: Spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orientifold with discrete torsion and $(\mu_3, \mu_1) = (+1, +1)$. The four sets of $D5_{1,n}$-branes are located at the four $\mathbb{Z}_4$ fixed points in the second and third torus. The notation is explained in the tables of section 4.1.

equations are incompatible, i.e. there is no possible brane configuration that satisfies both of them. To see this, denote the $\mathbb{Z}_4$ fixed points by $n_{2/3} = 0, 2$ and the remaining two $\mathbb{Z}_2$ fixed points by $n_{2/3} = 1, 3$. Summing the two equations for the fixed points $(0, 1)$ and $(1, 0)$ and subtracting the equation for the fixed point $(1, 1)$, one finds $\mathrm{Tr}(\gamma_{5_2,0}) + \mathrm{Tr}(\gamma_{5_3,0}) = 0$ which contradicts the second line in (4.3). To derive this result, it was crucial that $\mathrm{Tr}(\gamma_{(0,2),5_2})$ does only depend on n_3 but not on n_2. This is no longer true if Wilson lines in the second complex plane are added. Probably, there is a solution to the tadpole conditions if appropriate Wilson lines in the second and third complex plane are added.

As we will see, a similar problem arises in the $\mathbb{Z}_4 \times \mathbb{Z}_4$ orientifold. Indeed, it is easy to see that a $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifold with $N = 4$ and/or $M = 4$ is only consistent if its $\mathbb{Z}_4$ has vector structure.

4.3 $\mathbb{Z}_4 \times \mathbb{Z}_4$, $v = \frac{1}{4}(1, -1, 0)$, $w = \frac{1}{4}(0, 1, -1)$

This orientifold is symmetric under a permutation of the three sets of $D5_i$-branes. Thus, only four of the eight possible $\mathbb{Z}_4 \times \mathbb{Z}_4$ models are inequivalent. However, only for the models with vector structure in each of the two $\mathbb{Z}_4$ factors, i.e. $\mu_3 = \mu_1 = +1$,

$\mathbb{Z}_2 \times \mathbb{Z}_4,\ (\alpha_1,\alpha_2,\alpha_3) = (-1,+1,+1),\ \epsilon = -1$	
open string spectrum	
sector	gauge group / matter fields
99 (susy)	$\mathrm{SO}(8)_1 \times \mathrm{USp}(8)_2$ $2\,(\square_1,\square_2),\ \begin{smallmatrix}\square\\\square\end{smallmatrix}_1,\ \begin{smallmatrix}\square\\\square\end{smallmatrix}_2$
$5_{1,n}5_{1,n},\ n=0,1,2,3$ (non-susy)	$\mathrm{USp}(4)$ spinors: $2\begin{smallmatrix}\square\\\square\end{smallmatrix}$, scalars: $\square\!\square$
$5_25_2,\ 5_35_3$ (susy)	$\mathrm{U}(8)$ $adj,\ 2\begin{smallmatrix}\square\\\square\end{smallmatrix},\ \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}},\ \overline{\square\!\square}$
$95_{1,n}$ (non-susy)	spinors: $(\square_1,\square)$, scalars: $(\square_2,\square)$
$95_2,\ 95_3$ (susy)	$(\square_1,\square),\ (\square_2,\overline{\square})$
5_25_3 (susy)	$(\square,\square),\ (\overline{\square},\overline{\square})$
$5_35_{1,n}$ (non-susy)	spinors: $(\square,\square)$, scalars: $(\square,\square)$
$5_{1,n}5_2$ (non-susy)	spinors: $(\square,\square)$, scalars: $(\square,\overline{\square})$
closed string spectrum	
sector	$\mathcal{N}=1$ multiplets
untw.	gravity, 1 lin., 4 chir.
order-two	10 lin., 8 chir.
remaining	4 chir., 4 vec.

Table 8: Spectrum of the $\mathbb{Z}_2\times\mathbb{Z}_4$ orientifold with discrete torsion and $(\mu_3,\mu_1)=(-1,+1)$. The four sets of $\mathrm{D}5_{1,n}$-branes are located at the four $\mathbb{Z}_4$ fixed points in the second and third torus. The notation is explained in the tables of section 4.1.

does a consistent solution to the tadpole equations exist, at least in the absence of Wilson lines. From (3.18), we find that $\alpha_1\alpha_2\alpha_3 = -1$. Thus, only an odd number of negative α_i's is allowed. We found a solution for $(-1,-1,-1)$ with and without discrete torsion.

The only tadpole conditions with non-vanishing Klein bottle contribution are (3.19) and (3.23), corresponding to the untwisted sector and the the three order-two sectors $\bar{k}_1=(0,2)$, $\bar{k}_2=(2,2)$, $\bar{k}_3=(2,0)$. The untwisted tadpoles fix the rank of the gauge group to be 8 (16) for each of the four sectors 99, 5_i5_i in the case with (without) discrete torsion. For the $(-1,-1,-1)$ model, the tadpole conditions of the $\bar{k}_i$ sector read

$$\begin{aligned}\mathrm{Tr}(\gamma_{\bar{k}_i,9}) = 0\,, \qquad \mathrm{Tr}(\gamma_{\bar{k}_i,5_i,n_i}) = \epsilon^{-k_{i,1}k_{i,2}/4}\,8\,,\\ \mathrm{Tr}(\gamma_{\bar{k}_i,5_j,n_i}) = \mathrm{Tr}(\gamma_{\bar{k}_i,5_k,n_i}) = 0\,,\end{aligned} \tag{4.4}$$

where $k_{i,1}$, $k_{i,2}$ are the components of the two-vector $\bar{k}_i$ and $n_i=0,1,2,3$ denotes the $\mathbb{Z}_4$ fixed points in the j-th and k-th complex plane.[13] We assumed that there

[13]Here, (ijk) is a permutation of (123).

$\mathbb{Z}_2 \times \mathbb{Z}_4,\ (\alpha_1, \alpha_2, \alpha_3) = (-1,-1,-1),\ \epsilon = 1$	
open string spectrum	
sector	gauge group / matter fields
99 (susy)	$\mathrm{SO}(4)_1 \times \mathrm{U}(4)_2 \times \mathrm{SO}(4)_3 \times \mathrm{SO}(4)_4 \times \mathrm{U}(4)_5 \times \mathrm{SO}(4)_6$ $(\square_1, \square_2),\ (\square_1, \square_4),\ (\square_1, \overline{\square}_5),\ (\overline{\square}_2, \square_3),\ (\overline{\square}_2, \square_4),\ (\square_2, \overline{\square}_5),$ $(\square_2, \square_6),\ (\square_3, \square_5),\ (\square_3, \square_6),\ (\square_4, \square_5),\ (\square_5, \overline{\square}_2),\ (\overline{\square}_5, \square_6)$
$5_{1,n}5_{1,n}$ $n = 0,1,2,3$ (non-susy)	$\mathrm{U}(2)_1 \times \mathrm{U}(2)_2$ spinors: $adj_1,\ adj_2,\ \genfrac{}{}{0pt}{}{\square}{\square}_1,\ \overline{\genfrac{}{}{0pt}{}{\square}{\square}}_1,\ \genfrac{}{}{0pt}{}{\square}{\square}_2,\ \overline{\genfrac{}{}{0pt}{}{\square}{\square}}_2$ scalars: $\square\!\square_1,\ \overline{\square\!\square}_1,\ \square\!\square_2,\ \overline{\square\!\square}_2$
$5_2 5_2,\ 5_3 5_3$ (non-susy)	$\mathrm{U}(4)_1 \times \mathrm{U}(4)_2 \times \mathrm{U}(4)_3 \times \mathrm{U}(4)_4$ $\mathcal{N} = 1$ mult.: $(\square_1, \square_3),\ (\square_1, \square_4),\ (\square_1, \overline{\square}_4),\ (\overline{\square}_1, \overline{\square}_4),\ (\square_2, \overline{\square}_1),$ $(\square_2, \square_3),\ (\overline{\square}_2, \overline{\square}_3),\ (\overline{\square}_2, \overline{\square}_4),\ (\square_3, \overline{\square}_2),\ (\square_4, \overline{\square}_3)$ spinors: $adj_1,\ adj_2,\ adj_3,\ adj_4,\ \overline{\genfrac{}{}{0pt}{}{\square}{\square}}_1,\ \genfrac{}{}{0pt}{}{\square}{\square}_2,\ \overline{\genfrac{}{}{0pt}{}{\square}{\square}}_3,\ \genfrac{}{}{0pt}{}{\square}{\square}_4$ scalars: $\overline{\square\!\square}_1,\ \square\!\square_2,\ \overline{\square\!\square}_3,\ \square\!\square_4$
$95_{1,n}$ (non-susy)	spinors: $(\square_1, \overline{\square}_1),\ (\square_3, \overline{\square}_2),\ (\square_4, \square_1),\ (\square_6, \square_2)$ scalars: $(\square_2, \overline{\square}_1),\ (\overline{\square}_2, \overline{\square}_2),\ (\square_5, \square_1),\ (\overline{\square}_5, \square_2)$
95_2 (non-susy)	spinors: $(\square_1, \square_4),\ (\overline{\square}_2, \square_1),\ (\square_2, \square_3),\ (\square_3, \square_2),$ $(\square_4, \overline{\square}_1),\ (\square_5, \overline{\square}_2),\ (\overline{\square}_5, \overline{\square}_4),\ (\square_6, \overline{\square}_3)$ scalars: $(\square_1, \overline{\square}_1),\ (\square_2, \overline{\square}_2),\ (\overline{\square}_2, \overline{\square}_4),\ (\square_3, \overline{\square}_3),$ $(\square_4, \square_4),\ (\overline{\square}_5, \square_1),\ (\square_5, \square_3),\ (\square_6, \square_2)$
95_3 (non-susy)	spinors: $(\square_1, \overline{\square}_1),\ (\square_2, \square_1),\ (\overline{\square}_2, \square_3),\ (\square_3, \overline{\square}_3),$ $(\square_4, \square_4),\ (\square_5, \overline{\square}_4),\ (\overline{\square}_5, \overline{\square}_2),\ (\square_6, \square_2)$ scalars: $(\square_1, \overline{\square}_4),\ (\overline{\square}_2, \square_4),\ (\square_2, \square_2),\ (\square_3, \overline{\square}_2),$ $(\square_4, \square_1),\ (\overline{\square}_6, \overline{\square}_1),\ (\square_6, \overline{\square}_3),\ (\square_6, \square_3)$
$5_2 5_3$ (susy)	$(\square_1, \square_1),\ (\overline{\square}_1, \square_4),\ (\square_2, \overline{\square}_3),\ (\overline{\square}_2, \overline{\square}_2),$ $(\square_3, \square_3),\ (\overline{\square}_3, \square_2),\ (\square_4, \overline{\square}_1),\ (\overline{\square}_4, \overline{\square}_4)$
$5_3 5_{1,n}$ (susy)	$(\overline{\square}_1, \overline{\square}_1),\ (\overline{\square}_3, \overline{\square}_2),\ (\square_4, \square_1),\ (\square_2, \square_2)$
$5_{1,n} 5_2$ (susy)	$(\square_1, \overline{\square}_1),\ (\overline{\square}_1, \square_4),\ (\overline{\square}_3, \square_2),\ (\square_3, \overline{\square}_3),$
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 4 chir.
order-two	34 lin.
remaining	20 lin., 4 chir.

Table 9: Spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orientifold without discrete torsion and $(\mu_3, \mu_1) = (+1, +1)$. The four sets of $\mathrm{D}5_{1,n}$-branes are located at the four $\mathbb{Z}_4$ fixed points in the second and third torus. The notation is explained in the tables of section 4.1.

are no $\mathrm{D}5_i$-branes at the 12 remaining $\mathbb{Z}_2$ fixed points. The sign $\epsilon^{-k_{i,1}k_{i,2}/4}$ is -1 for the $(2,2)$ sector in the model with discrete torsion and $+1$ else. The conditions (4.4) were obtained from (3.23) by analysing the contributions of the $\mathrm{D}5_i$-branes to the 16

$\mathbb{Z}_2 \times \mathbb{Z}_4,\ (\alpha_1, \alpha_2, \alpha_3) = (-1, +1, +1),\ \epsilon = 1$	
open string spectrum	
sector	gauge group / matter fields
99 (susy)	$\mathrm{U}(4)_1 \times \mathrm{U}(4)_2 \times \mathrm{U}(4)_3 \times \mathrm{U}(4)_4$ $(\overline{\Box}_1, \overline{\Box}_2),\ (\Box_1, \Box_4),\ (\Box_1, \overline{\Box}_4),\ (\Box_2, \overline{\Box}_1),\ (\Box_2, \Box_3), (\Box_2, \Box_4),$ $(\overline{\Box}_2, \overline{\Box}_4),\ (\Box_3, \overline{\Box}_2),\ (\overline{\Box}_3, \overline{\Box}_4),\ (\Box_4, \overline{\Box}_3),\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_1,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_1,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_3,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_3$
$5_{1,n}5_{1,n}$ $n = 0, 1, 2, 3$ (non-susy)	$\mathrm{USp}(2)_1 \times \mathrm{USp}(2)_2 \times \mathrm{USp}(2)_3 \times \mathrm{USp}(2)_4$ $\mathcal{N} = 1$ mult.: $(\Box_1, \Box_3),\ (\Box_2, \Box_4)$ spinors: $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_1,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_2,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_3,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_4$
$5_25_2,\ 5_35_3$ (susy)	$\mathrm{U}(4)_1 \times \mathrm{U}(4)_2 \times \mathrm{U}(4)_3 \times \mathrm{U}(4)_4$ $(\Box_1, \overline{\Box}_3),\ (\overline{\Box}_1, \overline{\Box}_3),\ (\Box_1, \overline{\Box}_4),\ (\Box_2, \overline{\Box}_1),\ (\Box_2, \Box_4),\ (\Box_2, \overline{\Box}_4),$ $(\Box_3, \overline{\Box}_1),\ (\Box_3, \overline{\Box}_2),\ (\Box_4, \overline{\Box}_2),\ (\Box_4, \overline{\Box}_3),\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_1,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_2,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_3,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_4$
$95_{1,n}$ (non-susy)	spinors: $(\overline{\Box}_1, \Box_1),\ (\Box_1, \Box_3),\ (\overline{\Box}_3, \Box_2),\ (\Box_3, \Box_4)$ scalars: $(\overline{\Box}_2, \Box_2),\ (\Box_2, \Box_3),\ (\overline{\Box}_4, \Box_1),\ (\Box_4, \Box_4)$
95_2 (susy)	$(\overline{\Box}_1, \Box_1),\ (\Box_1, \Box_3),\ (\overline{\Box}_2, \Box_2),\ (\Box_2, \overline{\Box}_3),$ $(\overline{\Box}_3, \overline{\Box}_2),\ (\Box_3, \overline{\Box}_4),\ (\overline{\Box}_4, \overline{\Box}_1),\ (\Box_4, \Box_4)$
95_3 (susy)	$(\Box_1, \Box_1),\ (\overline{\Box}_1, \Box_3),\ (\overline{\Box}_2, \overline{\Box}_1),\ (\Box_2, \Box_4),$ $(\overline{\Box}_3, \overline{\Box}_4),\ (\Box_3, \overline{\Box}_2),\ (\overline{\Box}_4, \Box_2),\ (\Box_4, \overline{\Box}_3)$
5_25_3 (susy)	$(\overline{\Box}_1, \overline{\Box}_1),\ (\Box_1, \Box_3),\ (\overline{\Box}_2, \overline{\Box}_4),\ (\Box_2, \Box_2),$ $(\Box_3, \Box_1),\ (\overline{\Box}_3, \overline{\Box}_3),\ (\Box_4, \Box_4),\ (\overline{\Box}_4, \overline{\Box}_2)$
$5_35_{1,n}$ (non-susy)	spinors: $(\Box_1, \Box_3),\ (\overline{\Box}_4, \Box_2),\ (\overline{\Box}_2, \Box_4),\ (\Box_3, \Box_1)$ scalars: $(\Box_1, \Box_1),\ (\overline{\Box}_4, \Box_4),\ (\overline{\Box}_2, \Box_2),\ (\Box_3, \Box_3)$
$5_{1,n}5_2$ (non-susy)	spinors: $(\Box_1, \Box_1),\ (\Box_2, \overline{\Box}_2),\ (\Box_3, \Box_3),\ (\Box_4, \overline{\Box}_4)$ scalars: $(\Box_1, \overline{\Box}_3),\ (\Box_2, \Box_4),\ (\Box_3, \overline{\Box}_1),\ (\Box_4, \Box_2)$
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 4 chir.
order-two	10 lin., 24 chir.
remaining	20 lin., 4 chir.

Table 10: Spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orientifold without discrete torsion and $(\mu_3, \mu_1) = (-1, +1)$. The four sets of $\mathrm{D}5_{1,n}$-branes are located at the four $\mathbb{Z}_4$ fixed points in the second and third torus. The notation is explained in the tables of section 4.1.

fixed points and using the arguments given at the end of section 4.1. Generalising the argument showing that in the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orientifold $\gamma_{(0,2),5_2}$ is traceless, one can verify that in the present case the matrices $\gamma_{\bar{k}_i,5_j}$ are traceless for $i \neq j$. Therefore the last two conditions in (4.4) are trivially satisfied. The second condition in (4.4) forces us to put some $\mathrm{D}5_i$-branes at each of the four $\mathbb{Z}_4$ fixed points in the j-th and k-th complex plane. In the following, we will consider the most symmetric situation where eight $\mathrm{D}5_i$-branes sit at each $\mathbb{Z}_4$ fixed point. This leads to a unique solution of the tadpole equations. The complete spectrum is displayed in tables 11–13.

$\mathbb{Z}_4 \times \mathbb{Z}_4,\ (\alpha_1, \alpha_2, \alpha_3) = (-1, -1, -1),\ \epsilon = -1$	
open string spectrum	
sector	gauge group / matter fields
99 (susy)	$\mathrm{SO}(4)_1 \times \mathrm{SO}(4)_2 \times \mathrm{SO}(4)_3 \times \mathrm{USp}(4)_4$ $(\Box_1, \Box_2),\ (\Box_1, \Box_3),\ (\Box_1, \Box_4),\ (\Box_2, \Box_3),\ (\Box_2, \Box_4),\ (\Box_3, \Box_4)$
$5_{i,n}5_{i,n}$ (non-susy)	$\mathrm{U}(2)$ spinors: $adj,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix},\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}},\quad$ scalars: $\Box\!\Box,\ \overline{\Box\!\Box}$
$95_{1,n}$ (non-susy)	spinors: $(\Box_1, \overline{\Box}),\ (\Box_3, \Box)$ scalars: $(\Box_2, \overline{\Box}),\ (\Box_4, \Box)$
$95_{2,n}$ (non-susy)	spinors: $(\Box_2, \Box),\ (\Box_3, \overline{\Box})$ scalars: $(\Box_1, \overline{\Box}),\ (\Box_4, \Box)$
$95_{3,n}$ (non-susy)	spinors: $(\Box_1, \Box),\ (\Box_2, \overline{\Box})$ scalars: $(\Box_3, \overline{\Box}),\ (\Box_4, \Box)$
$5_{i,n}5_{j,m}$ (susy)	$(\overline{\Box}, \Box)$
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 3 chir.
order-two	27 lin.
remaining	12 lin.

Table 11: Spectrum of the $\mathbb{Z}_4\times\mathbb{Z}_4$ orientifold with discrete torsion and $(\mu_3, \mu_1) = (+1, +1)$. The 12 sets of D$5_{i,n}$-branes, $i = 1, 2, 3$, $n = 0, 1, 2, 3$, are located at the four $\mathbb{Z}_4$ fixed points in the j-th and k-th torus, where (ijk) is a permutation of (123). The matter in the $5_{i,n}5_{j,m}$ sectors is only present if the fixed points n and m are located at the same point in the k-th torus. The notation is explained in the tables of section 4.1.

$\mathbb{Z}_4 \times \mathbb{Z}_4,\ (\alpha_1, \alpha_2, \alpha_3) = (-1,-1,-1),\ \epsilon = 1$	
open string spectrum	
sector	gauge group / matter fields
99 (susy)	$\mathrm{SO}(2)_1 \times \mathrm{U}(2)_2 \times \mathrm{SO}(2)_3 \times \mathrm{U}(2)_4 \times \mathrm{U}(2)_5 \times \mathrm{U}(2)_6$ $\times\, \mathrm{U}(2)_7 \times \mathrm{SO}(2)_8 \times \mathrm{U}(2)_9 \times \mathrm{SO}(2)_{10}$ $(\square_1,\square_2),\ (\square_1,\overline{\square}_4),\ (\square_1,\square_7),\ (\overline{\square}_2,\square_3),\ (\square_2,\overline{\square}_5),\ (\square_2,\square_6),$ $(\overline{\square}_2,\overline{\square}_7),\ (\square_3,\square_5),\ (\square_3,\overline{\square}_6),\ (\square_4,\overline{\square}_2),\ (\square_4,\overline{\square}_7),\ (\square_4,\square_8),$ $(\overline{\square}_4,\overline{\square}_9),\ (\square_5,\overline{\square}_4),\ (\overline{\square}_5,\square_8),\ (\square_5,\overline{\square}_9),\ (\square_6,\overline{\square}_5),\ (\square_6,\square_{10}),$ $(\square_7,\overline{\square}_6),\ (\square_7,\square_9),\ (\overline{\square}_7,\square_{10}),\ (\square_8,\square_9),\ (\square_9,\overline{\square}_6),\ (\overline{\square}_9,\square_{10})$
$5_{i,n}5_{i,n}$ (non-susy)	$\mathrm{U}(1)_1 \times \mathrm{U}(1)_2 \times \mathrm{U}(1)_3 \times \mathrm{U}(1)_4$ $\mathcal{N}=1$ mult.: $(\square_1,\overline{\square}_3),\ (\square_2,\overline{\square}_4)$ spinors: 4 singlets, scalars: $\overline{\square\!\square}_1,\ \overline{\square\!\square}_2,\ \square\!\square_3,\ \square\!\square_4$
$95_{1,n}$ (non-susy)	spinors: $(\square_1,\overline{\square}_1),\ (\square_3,\overline{\square}_2),\ (\overline{\square}_4,\square_1),\ (\square_4,\overline{\square}_3),$ $(\overline{\square}_6,\square_2),\ (\square_6,\overline{\square}_4),\ (\square_8,\square_3),\ (\square_{10},\square_4)$ scalars: $(\square_2,\overline{\square}_1),\ (\overline{\square}_2,\overline{\square}_2),\ (\overline{\square}_5,\square_2),\ (\square_5,\overline{\square}_3),$ $(\overline{\square}_7,\square_1),\ (\square_7,\overline{\square}_4),\ (\square_9,\square_3),\ (\overline{\square}_9,\square_4)$
$95_{2,n}$ (non-susy)	spinors: $(\square_1,\square_4),\ (\square_3,\square_3),\ (\square_5,\overline{\square}_3),\ (\overline{\square}_5,\square_1),$ $(\square_7,\overline{\square}_4),\ (\overline{\square}_7,\square_2),\ (\square_8,\overline{\square}_1),\ (\square_{10},\overline{\square}_2)$ scalars: $(\square_2,\overline{\square}_3),\ (\overline{\square}_2,\overline{\square}_4),\ (\overline{\square}_4,\square_4),\ (\square_4,\overline{\square}_1),$ $(\overline{\square}_6,\square_3),\ (\square_6,\overline{\square}_2),\ (\overline{\square}_9,\square_1),\ (\square_9,\square_2)$
$95_{3,n}$ (non-susy)	spinors: $(\square_1,\square_4),\ (\square_2,\overline{\square}_4),\ (\overline{\square}_2,\square_2),\ (\square_3,\overline{\square}_2),$ $(\square_8,\square_3),\ (\square_9,\overline{\square}_3),\ (\overline{\square}_9,\square_1),\ (\square_{10},\overline{\square}_1)$ scalars: $(\square_4,\overline{\square}_4),\ (\overline{\square}_4,\overline{\square}_3),\ (\square_5,\overline{\square}_2),\ (\overline{\square}_5,\square_3),$ $(\square_6,\square_2),\ (\overline{\square}_6,\square_1),\ (\square_7,\square_4),\ (\overline{\square}_7,\overline{\square}_1)$
$5_{2,n}5_{3,m}$ (susy)	$(\square_3,\overline{\square}_2),\ (\square_4,\square_4),\ (\overline{\square}_1,\square_3),\ (\overline{\square}_2,\overline{\square}_1)$
$5_{3,n}5_{1,m}$ (susy)	$(\square_4,\overline{\square}_1),\ (\overline{\square}_2,\overline{\square}_2),\ (\square_3,\square_3),\ (\overline{\square}_1,\square_4)$
$5_{1,n}5_{2,m}$ (susy)	$(\overline{\square}_1,\square_4),\ (\overline{\square}_2,\square_3),\ (\square_3,\overline{\square}_1),\ (\square_4,\overline{\square}_2)$

Table 12: Spectrum of the $\mathbb{Z}_4 \times \mathbb{Z}_4$ orientifold without discrete torsion and $(\mu_3, \mu_1) = (+1,+1)$. There are 12 sets of D$5_{i,n}$-branes, $i = 1,2,3$, $n = 0,1,2,3$, as in the model of table 11. The matter in the $5_{i,n}5_{j,m}$ sectors is only present if the fixed points n and m are located at the same point in the k-th torus. The symmetric tensors in the $5_{i,n}5_{i,n}$ sectors are fields that carry double U(1) charge.

$\mathbb{Z}_4 \times \mathbb{Z}_4,\ (\alpha_1, \alpha_2, \alpha_3) = (-1, -1, -1),\ \epsilon = 1$	
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 3 chir.
order-two	27 lin.
remaining	48 lin., 12 chir.

Table 13: Closed string spectrum of the $\mathbb{Z}_4 \times \mathbb{Z}_4$ orientifold without discrete torsion and $(\mu_3, \mu_1) = (+1, +1)$.

$\mathbb{Z}_2 \times \mathbb{Z}_6,\ (\alpha_1, \alpha_2, \alpha_3) = (+1, +1, +1),\ \epsilon = -1$	
open string spectrum	
sector	gauge group / matter fields
$99,\ 5_i5_i$ (susy)	$U(4)_1 \times USp(8)_2$ $(\Box_1, \Box_2),\ (\overline{\Box}_1, \Box_2),\ adj_1,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_1,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_1,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_2$
$95_1,\ 5_25_3$ (susy)	$(\Box_1, \overline{\Box}_1),\ (\overline{\Box}_1, \Box_1),\ (\Box_2, \Box_2)$
$95_2,\ 5_15_3$ (susy)	$(\Box_1, \Box_2),\ (\overline{\Box}_1, \Box_1),\ (\Box_2, \overline{\Box}_1)$
$95_3,\ 5_15_2$ (susy)	$(\overline{\Box}_1, \Box_2),\ (\Box_1, \overline{\Box}_1),\ (\Box_2, \Box_1)$
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 4 chir.
order-two	14 chir.
remaining	12 lin., 6 chir., 2 vec.

Table 14: Spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_6$ orientifold with discrete torsion and $(\mu_3, \mu_1) = (-1, -1)$. The notation is explained in the tables of section 4.1.

$\mathbb{Z}_2\times\mathbb{Z}_6,\ (\alpha_1,\alpha_2,\alpha_3)=(-1,+1,-1),\ \epsilon=-1$	
open string spectrum	
sector	gauge group / matter fields
$99,\ 5_2 5_2$ (susy)	$\mathrm{SO}(8)_1\times \mathrm{U}(4)_2$ $(\Box_1,\Box_2),\ (\Box_1,\overline{\Box}_2),\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_1,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_2,\ \Box\!\Box_2,\ adj_2$
$5_1 5_1,\ 5_3 5_3$ (non-susy)	$\mathrm{USp}(8)_1\times \mathrm{U}(4)_2$ $\mathcal{N}=1$ mult.: $(\Box_1,\Box_2),\ (\Box_1,\overline{\Box}_2)$ spinors: $2\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_1,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_2,\ \Box\!\Box_2,\ 2\,adj_2,$ scalars: $\Box\!\Box_1,\ \overline{\Box\!\Box}_2,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_2,\ adj_2,$
$95_1,\ 5_2 5_3$ (non-susy)	spinors: $(\Box_1,\Box_1),\ (\Box_2,\overline{\Box}_2),\ (\overline{\Box}_2,\Box_2)$ scalars: $(\Box_1,\Box_2),\ (\Box_2,\Box_1),\ (\overline{\Box}_2,\overline{\Box}_2)$
$95_2,\ 5_3 5_1$ (susy)	$(\Box_1,\overline{\Box}_2),\ (\Box_2,\Box_2),\ (\overline{\Box}_2,\Box_1)$
$95_3,\ 5_1 5_2$ (non-susy)	spinors: $(\Box_1,\overline{\Box}_2),\ (\Box_2,\Box_2),\ (\overline{\Box}_2,\Box_1)$ scalars: $(\Box_1,\Box_2),\ (\overline{\Box}_2,\overline{\Box}_2),\ (\Box_2,\Box_1)$
closed string spectrum	
sector	$\mathcal{N}=1$ multiplets
untw.	gravity, 1 lin., 4 chir.
order-two	4 chir., 10 vec.
remaining	12 lin., 6 chir., 2 vec.

Table 15: Spectrum of the $\mathbb{Z}_2\times\mathbb{Z}_6$ orientifold with discrete torsion and $(\mu_3,\mu_1)=(+1,+1)$. The notation is explained in the tables of section 4.1.

JHEP10(2000)049

$\mathbb{Z}_2 \times \mathbb{Z}_6,\ (\alpha_1, \alpha_2, \alpha_3) = (+1, -1, -1),\ \epsilon = -1$	
open string spectrum	
sector	gauge group / matter fields
$99,\ 5_1 5_1$ (susy)	$\mathrm{SO}(8)_1 \times \mathrm{U}(4)_2$ $(\Box_1, \Box_2),\ (\Box_1, \overline{\Box}_2),\ \Box\!\Box_1,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_2,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_2,\ adj_2$
$5_2 5_2,\ 5_3 5_3$ (non-susy)	$\mathrm{USp}(8)_1 \times \mathrm{U}(4)_2$ $\mathcal{N} = 1$ mult.: $(\Box_1, \Box_2),\ (\Box_1, \overline{\Box}_2)$ spinors: $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_1,\ \Box\!\Box_1,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_2,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_2,\ 2\,adj_2$ scalars: $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_1,\ \Box\!\Box_2,\ \overline{\Box\!\Box}_2,\ adj_2$
$95_1,\ 5_2 5_3$ (susy)	$(\Box_1, \Box_1),\ (\Box_2, \overline{\Box}_2),\ (\overline{\Box}_2, \Box_2)$
$95_2,\ 5_1 5_3$ (non-susy)	spinors: $(\Box_1, \Box_2),\ (\overline{\Box}_2, \Box_1),\ (\Box_2, \overline{\Box}_2)$ scalars: $(\Box_1, \Box_2),\ (\overline{\Box}_2, \Box_1),\ (\Box_2, \overline{\Box}_2)$
$95_3,\ 5_1 5_2$ (non-susy)	spinors: $(\Box_1, \overline{\Box}_2),\ (\Box_2, \Box_1),\ (\overline{\Box}_2, \Box_2)$ scalars: $(\Box_1, \Box_2),\ (\overline{\Box}_2, \Box_1),\ (\Box_2, \overline{\Box}_2)$
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 4 chir.
order-two	6 chir., 8 vec.
remaining	12 lin., 6 chir., 2 vec.

Table 16: Spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_6$ orientifold with discrete torsion and $(\mu_3, \mu_1) = (+1, -1)$. The notation is explained in the tables of section 4.1.

$\mathbb{Z}_2 \times \mathbb{Z}_6,\ (\alpha_1,\alpha_2,\alpha_3) = (-1,-1,-1),\ \epsilon = 1$	
open string spectrum	
sector	gauge group / matter fields
99 (susy)	$\mathrm{SO}(8-2t_1)_1 \times U(4-t_1)_2 \times U(t_1)_3 \times \mathrm{SO}(2t_1)_4$ $\times\, \mathrm{SO}(8-2t_1)_5 \times U(4-t_1)_6 \times U(t_1)_7 \times \mathrm{SO}(2t_1)_8$ $(\square_1,\square_2),\ (\square_1,\square_5),\ (\square_1,\overline{\square}_6),\ (\overline{\square}_2,\square_5),\ (\square_2,\overline{\square}_6),\ (\square_2,\overline{\square}_7),$ $(\square_3,\overline{\square}_2),\ (\overline{\square}_3,\square_4),\ (\square_3,\overline{\square}_7),\ (\square_3,\square_8),\ (\square_4,\square_7),\ (\square_4,\square_8),$ $(\square_5,\square_6),\ (\square_6,\overline{\square}_2),\ (\square_6,\overline{\square}_3),\ (\square_7,\overline{\square}_3),\ (\square_7,\overline{\square}_6),\ (\overline{\square}_7,\square_8)$
$5_1 5_1$ (non-susy)	$U(2t_1)_1 \times U(t_1)_2 \times U(4-t_1)_3 \times U(8-2t_1)_4 \times U(4-t_1)_5 \times U(t_1)_6$ $\mathcal{N}=1$ mult.: $(\overline{\square}_1,\overline{\square}_2),\ (\square_1,\square_6),\ (\square_1,\overline{\square}_6),\ (\square_2,\overline{\square}_1),\ (\square_2,\square_5),\ (\square_2,\square_6),$ $(\overline{\square}_2,\overline{\square}_6),\ (\square_3,\overline{\square}_2),\ (\square_3,\square_4),\ (\square_3,\square_5),\ (\overline{\square}_3,\overline{\square}_5),\ (\overline{\square}_3,\overline{\square}_6),$ $(\square_4,\overline{\square}_3),\ (\overline{\square}_4,\overline{\square}_5),\ (\square_5,\overline{\square}_4),\ (\square_6,\overline{\square}_5)$ spinors: $adj_1,\ \ldots,\ adj_6,\ {\substack{\square\\\square}}_1,\ \overline{\substack{\square\\\square}}_1,\ {\substack{\square\\\square}}_4,\ \overline{\substack{\square\\\square}}_4$ scalars: $\square\square_1,\ \overline{\square\square}_1,\ \square\square_4,\ \overline{\square\square}_4$
$5_2 5_2,\ 5_3 5_3$ (non-susy)	$U(4-t_2)_1 \times \mathrm{U}(4)_2 \times U(t_2)_3 \times U(t_2)_4 \times \mathrm{U}(4)_5 \times U(4-t_2)_6$ $(\square_1,\square_5),\ (\square_1,\square_6),\ (\square_1,\overline{\square}_6),\ (\overline{\square}_1,\overline{\square}_6),\ (\square_2,\overline{\square}_1),\ (\square_2,\square_4),$ $(\square_2,\square_5),\ (\overline{\square}_2,\overline{\square}_5),\ (\overline{\square}_2,\overline{\square}_6),\ (\square_3,\overline{\square}_2),\ (\square_3,\square_4),\ (\overline{\square}_3,\overline{\square}_4),$ $(\overline{\square}_3,\overline{\square}_5),\ (\square_4,\overline{\square}_3),\ (\square_5,\overline{\square}_4),\ (\square_6,\overline{\square}_5)$ spinors: $adj_1,\ \ldots,\ adj_6,\ \overline{\substack{\square\\\square}}_1,\ {\substack{\square\\\square}}_3,\ \overline{\substack{\square\\\square}}_4,\ {\substack{\square\\\square}}_6$ scalars: $\overline{\square\square}_1,\ \square\square_3,\ \overline{\square\square}_4,\ \square\square_6$
95_1 (non-susy)	spinors: $(\square_1,\overline{\square}_1),\ (\square_2,\overline{\square}_2),\ (\overline{\square}_2,\overline{\square}_6),\ (\square_3,\overline{\square}_3),\ (\overline{\square}_3,\overline{\square}_5),\ (\square_4,\overline{\square}_4),$ $(\square_5,\square_1),\ (\overline{\square}_6,\square_2),\ (\square_6,\square_6),\ (\overline{\square}_7,\square_3),\ (\square_7,\square_5),\ (\square_8,\square_4)$ scalars: $(\square_1,\overline{\square}_6),\ (\square_2,\overline{\square}_1),\ (\overline{\square}_2,\overline{\square}_5),\ (\square_3,\overline{\square}_2),\ (\overline{\square}_3,\overline{\square}_4),\ (\square_4,\overline{\square}_3),$ $(\square_5,\square_2),\ (\square_6,\square_1),\ (\overline{\square}_6,\square_3),\ (\overline{\square}_7,\square_4),\ (\square_7,\square_6),\ (\square_8,\square_5)$

Table 17: Spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_6$ orientifold without discrete torsion and $(\mu_3,\mu_1) = (+1,+1)$. There are 25 solutions to the tadpole equations, parametrised by $t_1, t_2 = 0,\ldots,4$. The notation is explained in the tables of section 4.1.

$\mathbb{Z}_2 \times \mathbb{Z}_6,\ (\alpha_1, \alpha_2, \alpha_3) = (-1, -1, -1),\ \epsilon = 1$	
open string spectrum	
sector	gauge group / matter fields
95_2 (non-susy)	spinors: $(\square_1,\square_6),\ (\overline{\square}_2,\square_1),\ (\square_2,\square_5),\ (\overline{\square}_3,\square_2),\ (\square_3,\square_4),\ (\square_4,\square_3),$ $(\square_5,\overline{\square}_1),\ (\square_6,\overline{\square}_2),\ (\overline{\square}_6,\overline{\square}_6),\ (\square_7,\overline{\square}_3),\ (\overline{\square}_7,\overline{\square}_5),\ (\square_8,\overline{\square}_4)$ scalars: $(\square_1,\overline{\square}_1),\ (\square_2,\overline{\square}_2),\ (\overline{\square}_2,\overline{\square}_6),\ (\square_3,\overline{\square}_3),\ (\overline{\square}_3,\overline{\square}_5),\ (\square_4,\overline{\square}_4),$ $(\square_5,\square_6),\ (\overline{\square}_6,\square_1),\ (\square_6,\square_5),\ (\overline{\square}_7,\square_2),\ (\square_7,\square_4),\ (\square_8,\square_3)$
95_3 (non-susy)	spinors: $(\square_1,\overline{\square}_1),\ (\square_2,\square_1),\ (\overline{\square}_2,\square_5),\ (\square_3,\overline{\square}_5),\ (\overline{\square}_3,\overline{\square}_3),\ (\square_4,\square_3),$ $(\square_5,\square_6),\ (\square_6,\overline{\square}_6), (\overline{\square}_6,\overline{\square}_2),\ (\square_7,\square_2),\ (\overline{\square}_7,\square_4),\ (\square_8,\overline{\square}_4)$ scalars: $(\square_1,\overline{\square}_6),\ (\overline{\square}_2,\square_6),\ (\square_2,\square_2),\ (\overline{\square}_3,\overline{\square}_2),\ (\square_3,\overline{\square}_4), (\square_4,\square_4),$ $(\square_5,\square_1),\ (\overline{\square}_6,\overline{\square}_1),\ (\square_6,\overline{\square}_5),\ (\overline{\square}_7,\square_5),\ (\square_7,\square_3),\ (\square_8,\overline{\square}_3)$
5_25_3 (susy)	$(\square_1,\square_1),\ (\overline{\square}_1,\square_6),\ (\square_2,\overline{\square}_5),\ (\overline{\square}_2,\overline{\square}_2),\ (\square_3,\square_3),\ (\overline{\square}_3,\square_4),$ $(\square_4,\overline{\square}_3),\ (\overline{\square}_4,\overline{\square}_4),\ (\square_5,\square_5),\ (\overline{\square}_5,\square_2),\ (\square_6,\overline{\square}_1),\ (\overline{\square}_6,\overline{\square}_6)$
5_35_1 (susy)	$(\overline{\square}_1,\overline{\square}_1),\ (\square_1,\overline{\square}_6),\ (\square_5,\overline{\square}_2),\ (\overline{\square}_5,\overline{\square}_5),\ (\overline{\square}_3,\overline{\square}_3),\ (\square_3,\overline{\square}_4),$ $(\square_6,\square_1),\ (\overline{\square}_6,\square_2),\ (\square_2,\square_3),\ (\overline{\square}_2,\square_6),\ (\overline{\square}_4,\square_4),\ (\square_4,\square_5)$
5_15_2 (susy)	$(\square_1,\overline{\square}_1),\ (\overline{\square}_1,\square_6),\ (\overline{\square}_2,\square_1),\ (\square_2,\overline{\square}_2),\ (\overline{\square}_3,\square_2),\ (\square_3,\overline{\square}_3),$ $(\overline{\square}_4,\square_3),\ (\square_4,\overline{\square}_4),\ (\overline{\square}_5,\square_4),\ (\square_5,\overline{\square}_5),\ (\overline{\square}_6,\square_5),\ (\square_6,\overline{\square}_6)$
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 4 chir.
order-two	22 lin.
remaining	21 lin., 6 chir., 1 vec.

Table 18: Spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_6$ orientifold without discrete torsion and $(\mu_3, \mu_1) = (+1, +1)$. (continued)

$\mathbb{Z}_2 \times \mathbb{Z}_6,\ (\alpha_1, \alpha_2, \alpha_3) = (+1, +1, -1),\ \epsilon = 1$	

open string spectrum	
sector	gauge group / matter fields
$99,\ 5_1 5_1$ (susy)	$U(4-t_1)_1 \times \mathrm{U}(4)_2 \times U(t_1)_3 \times U(t_1)_4 \times \mathrm{U}(4)_5 \times U(4-t_1)_6$ $(\Box_1, \Box_5),\ (\Box_1, \Box_6),\ (\Box_1, \overline{\Box}_6),\ (\overline{\Box}_1, \overline{\Box}_6),\ (\Box_2, \overline{\Box}_1),\ (\Box_2, \Box_4),$ $(\Box_2, \Box_5),\ (\overline{\Box}_2, \overline{\Box}_5),\ (\overline{\Box}_2, \overline{\Box}_6),\ (\Box_3, \overline{\Box}_2),\ (\Box_3, \Box_4),\ (\overline{\Box}_3, \overline{\Box}_4),$ $(\overline{\Box}_3, \overline{\Box}_5),\ (\Box_4, \overline{\Box}_3),\ (\Box_5, \overline{\Box}_4),\ (\Box_6, \overline{\Box}_5),\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_1,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_3,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_4,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_6$
$5_2 5_2$ (susy)	$U(8-2t_2)_1 \times U(4-t_2)_2 \times U(t_2)_3 \times U(2t_2)_4 \times U(t_2)_5 \times U(4-t_2)_6$ $(\overline{\Box}_1, \overline{\Box}_2),\ (\Box_1, \Box_6),\ (\Box_1, \overline{\Box}_6),\ (\Box_2, \overline{\Box}_1),\ (\Box_2, \Box_5),\ (\Box_2, \Box_6),$ $(\overline{\Box}_2, \overline{\Box}_6),\ (\Box_3, \overline{\Box}_2),\ (\Box_3, \Box_4),\ (\Box_3, \Box_5),\ (\overline{\Box}_3, \overline{\Box}_5),\ (\overline{\Box}_3, \overline{\Box}_6),$ $(\Box_4, \overline{\Box}_3),\ (\overline{\Box}_4, \overline{\Box}_5),\ (\Box_5, \overline{\Box}_4),\ (\Box_6, \overline{\Box}_5),\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_1,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_1,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_4,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_4$
$5_3 5_3$ (non-susy)	$\mathrm{USp}(2t_2)_1 \times U(t_2)_2 \times U(4-t_2)_3 \times \mathrm{USp}(8-2t_2)_4$ $\times\, \mathrm{USp}(2t_2)_5 \times U(t_2)_6 \times U(4-t_2)_7 \times \mathrm{USp}(8-2t_2)_8$ $\mathcal{N}=1$ mult.: $(\Box_1, \Box_2),\ (\Box_1, \Box_5),\ (\Box_1, \overline{\Box}_6),\ (\overline{\Box}_2, \Box_5),\ (\Box_2, \overline{\Box}_6),\ (\Box_2, \overline{\Box}_7),$ $(\Box_3, \overline{\Box}_2),\ (\overline{\Box}_3, \Box_4),\ (\Box_3, \overline{\Box}_7),\ (\Box_3, \Box_8),\ (\Box_4, \Box_7),\ (\Box_4, \Box_8),$ $(\Box_5, \Box_6),\ (\Box_6, \overline{\Box}_2),\ (\Box_6, \overline{\Box}_3),\ (\Box_7, \overline{\Box}_3),\ (\Box_7, \overline{\Box}_8),\ (\overline{\Box}_7, \Box_8)$ spinors: $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_1,\ adj_2,\ adj_3,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_4,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_5,\ adj_6,\ adj_7,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_8$
$95_i,\ 5_i 5_j$	many bifundamentals

closed string spectrum	
sector	$\mathcal{N}=1$ multiplets
untw.	gravity, 1 lin., 4 chir.
order-two	8 lin., 14 chir.
remaining	21 lin., 6 chir., 1 vec.

Table 19: Spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_6$ orientifold without discrete torsion and $(\mu_3, \mu_1) = (+1, -1)$. There are 25 solutions to the tadpole equations, parametrised by $t_1, t_2 = 0, \ldots, 4$. The notation is explained in the tables of section 4.1.

$\mathbb{Z}_2 \times \mathbb{Z}_6,\ (\alpha_1, \alpha_2, \alpha_3) = (-1, +1, +1),\ \epsilon = 1$	
open string spectrum	
sector	gauge group / matter fields
99 (susy)	$U(8-2t_1)_1 \times U(4-t_1)_2 \times U(t_1)_3 \times U(2t_1)_4 \times U(t_1)_5 \times U(4-t_1)_6$ $(\overline{\Box}_1, \overline{\Box}_2),\ (\Box_1, \Box_6),\ (\Box_1, \overline{\Box}_6),\ (\Box_2, \overline{\Box}_1),\ (\Box_2, \Box_5),\ (\Box_2, \Box_6),$ $(\overline{\Box}_2, \overline{\Box}_6),\ (\Box_3, \overline{\Box}_2),\ (\Box_3, \Box_4),\ (\Box_3, \Box_5),\ (\overline{\Box}_3, \overline{\Box}_5),\ (\overline{\Box}_3, \overline{\Box}_6),$ $(\Box_4, \overline{\Box}_3),\ (\overline{\Box}_4, \overline{\Box}_5),\ (\Box_5, \overline{\Box}_4),\ (\Box_6, \overline{\Box}_5),\ \boxminus_1,\ \overline{\boxminus}_1,\ \boxminus_4,\ \overline{\boxminus}_4$
$5_1 5_1$ (non-susy)	$\mathrm{USp}(2t_1)_1 \times U(t_1)_2 \times U(4-t_1)_3 \times \mathrm{USp}(8-2t_1)_4$ $\times\, \mathrm{USp}(2t_1)_5 \times U(t_1)_6 \times U(4-t_1)_7 \times \mathrm{USp}(8-2t_1)_8$ $\mathcal{N}=1$ mult.: $(\Box_1, \Box_2),\ (\Box_1, \Box_5),\ (\Box_1, \overline{\Box}_6),\ (\overline{\Box}_2, \Box_5),\ (\Box_2, \overline{\Box}_6),\ (\Box_2, \overline{\Box}_7),$ $(\Box_3, \overline{\Box}_2),\ (\overline{\Box}_3, \Box_4),\ (\Box_3, \overline{\Box}_7),\ (\Box_3, \Box_8),\ (\Box_4, \Box_7),\ (\Box_4, \Box_8),$ $(\Box_5, \Box_6),\ (\Box_6, \overline{\Box}_2),\ (\Box_6, \overline{\Box}_3),\ (\Box_7, \overline{\Box}_3),\ (\Box_7, \overline{\Box}_6),\ (\overline{\Box}_7, \Box_8)$ spinors: $\boxminus_1,\ adj_2,\ adj_3,\ \boxminus_4,\ \boxminus_5,\ adj_6,\ adj_7,\ \boxminus_8$
$5_2 5_2,\ 5_3 5_3$ (susy)	$U(t_2)_1 \times \mathrm{U}(4)_2 \times U(4-t_2)_3 \times U(4-t_2)_4 \times \mathrm{U}(4)_5 \times U(t_2)_6$ $(\Box_1, \Box_5),\ (\Box_1, \Box_6),\ (\Box_1, \overline{\Box}_6),\ (\overline{\Box}_1, \overline{\Box}_6),\ (\Box_2, \overline{\Box}_1),\ (\Box_2, \Box_4),$ $(\Box_2, \Box_5),\ (\overline{\Box}_2, \overline{\Box}_5),\ (\overline{\Box}_2, \overline{\Box}_6),\ (\Box_3, \overline{\Box}_2),\ (\Box_3, \Box_4),\ (\overline{\Box}_3, \overline{\Box}_4),$ $(\overline{\Box}_3, \overline{\Box}_5),\ (\Box_4, \overline{\Box}_3),\ (\Box_5, \overline{\Box}_4),\ (\Box_6, \overline{\Box}_5),\ \overline{\boxminus}_1,\ \boxminus_3,\ \overline{\boxminus}_4,\ \boxminus_6$
$95_i,\ 5_i 5_j$	many bifundamentals
closed string spectrum	
sector	$\mathcal{N}=1$ multiplets
untw.	gravity, 1 lin., 4 chir.
order-two	6 lin., 16 chir.
remaining	21 lin., 6 chir., 1 vec.

Table 20: Spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_6$ orientifold without discrete torsion and $(\mu_3, \mu_1) = (-1, +1)$. There are 25 solutions to the tadpole equations, parametrised by $t_1, t_2 = 0, \ldots, 4$. The notation is explained in the tables of section 4.1.

It is straightforward to generalise the argument for the impossibility of cancelling the RR twisted tadpoles in the $\mathbb{Z}_2 \times \mathbb{Z}_4$ $(+1,+1,-1)$ model. In the $\mathbb{Z}_4 \times \mathbb{Z}_4$ $(+1,+1,-1)$ model,[14] the incompatibility between the tadpole equations appears in in the sectors $(0,2)$ and $(2,2)$, independently of the value of discrete torsion. The problem appears in order-two sectors that fix a set of branes without vector structure. Only the $(-1,-1,-1)$ model with and without discrete torsion is found to be consistent.

4.4 $\mathbb{Z}_2 \times \mathbb{Z}_6$, $v = \frac{1}{2}(1,-1,0)$, $w = \frac{1}{6}(0,1,-1)$

This orientifold is not symmetric under an arbitrary permutation of the three sets of D5_i-branes. The only even twist, $\bar{k} = (0,2)$, fixes the world-volume of the D5_1-branes. As in the $\mathbb{Z}_2 \times \mathbb{Z}_4$ case, only the D5_2- and D5_3-branes appear on the same footing. Therefore, we expect that six of the eight possible $\mathbb{Z}_2 \times \mathbb{Z}_6$ models are inequivalent. The $(+1,+1,+1)$ model has already been constructed in [6].

Let us first concentrate on the cases with discrete torsion. The only non-trivial tadpole conditions are (3.19) and (3.23), corresponding to the untwisted sector and the $(0,2)$ sector. The former fixes the rank of the gauge group to be 8 for each of the four sectors 99, $5_i 5_i$. The latter read

$$\mathrm{Tr}(\gamma_{(0,2),9}) = \mathrm{Tr}(\gamma_{(0,2),5_1}) = -\alpha_1\, 8\,, \qquad \mathrm{Tr}(\gamma_{(0,2),5_2}) = \mathrm{Tr}(\gamma_{(0,2),5_3}) = \alpha_1\, 8\,. \qquad (4.5)$$

We assumed that all D5_i-branes are located at the origin. To see that the tadpole conditions (3.23) indeed simplify to the form (4.5), we need to consider the Klein bottle contributions to the various fixed points of the $(0,2)$ sector, using the method described at the end of section 3.3. Label the three $\mathbb{Z}_3$ fixed points in the second plane by $n_2 = 0,1,2$ and the three $\mathbb{Z}_3$ fixed points in the third plane $n_3 = 0,1,2$, where 0 denotes the origin. The Klein bottle contribution $\mathcal{K}_0$ (the term proportional to $c_2 c_3$ in (3.23)) is only present at the origin $(0,0)$, $\mathcal{K}_1$ (the term proportional to $\epsilon_1\,\beta_1\, s_2 s_3$) is present at all nine $\mathbb{Z}_3$ fixed points (n_2,n_3), $\mathcal{K}_2$ (the term proportional to $\epsilon_2\,\beta_2$) is present at the three fixed points $(0,n_3)$ and $\mathcal{K}_3$ (the term proportional to $\epsilon_3\,\beta_3$) is present at the three fixed points $(n_2,0)$. Using this and $\alpha_1 = \alpha_2\alpha_3$, it is easy to see that (3.23) reduces to (4.5). These equations have a unique solution. The results are displayed in tables 14–16.

In the cases without discrete torsion, the tadpole conditions for the $(0,2)$ sector are identical to eq. (4.5) if all D5_i-branes sit at the origin. Now, the remaining twisted sectors give additional conditions because, in general, the γ matrices are not traceless. The tadpole equations have 25 distinct solutions for each of the three possible models. We display the complete spectrum of the $(-1,-1,-1)$ model in tables 17 and 18 and restrict ourselves to give the 99 and $5_i 5_i$ spectrum of the two

[14] The three models $(+1,+1,-1)$, $(+1,-1,+1)$ and $(-1,+1,+1)$ are equivalent up to a permutation of the D5_i-branes.

remaining models in tables 19 and 20. We determined also the 95_i and 5_i5_j spectrum of the last two models and verified that the complete spectrum is free of non-abelian gauge anomalies.

4.5 $\mathbb{Z}_2 \times \mathbb{Z}_6'$, $v = \frac{1}{2}(1,-1,0)$, $w = \frac{1}{6}(-2,1,1)$

This orientifold is similar to the previous one. However, there are only four inequivalent models. This is because the only even twist, $\bar{k} = (0,2)$ has no fixed planes. Therefore, all three sets of $D5_i$-branes appear on the same footing. The $(+1,+1,+1)$ model has already been constructed in [8]. It is of some phenomenological interest because it contains a gauge group SU(6) with three generations of matter fields in the antisymmetric tensor representation.

The only non-trivial tadpole conditions for the models with discrete torsion are (3.19) and (3.21), corresponding to the untwisted sector and the $(0,2)$ sector. The former fixes the rank of the gauge group to be 8 for each of the four sectors 99, 5_i5_i. The latter reads

$$\mathrm{Tr}(\gamma_{(0,2),9}) = \mathrm{Tr}(\gamma_{(0,2),5_1}) = \alpha_1\, 4\,, \qquad \mathrm{Tr}(\gamma_{(0,2),5_2}) = \mathrm{Tr}(\gamma_{(0,2),5_3}) = -\alpha_1\, 4\,. \tag{4.6}$$

We assumed that all $D5_i$-branes are located at the origin. The 27 fixed points of the $(0,2)$ sector can be labelled by the triples (n_1,n_2,n_3), where $n_i = 0,1,2$ denotes the fixed points in the i-th complex plane and 0 is the origin. One finds that the Klein bottle contribution $\mathcal{K}_0$ is present at the three fixed points $(n_1,0,0)$, the contribution $\mathcal{K}_1$ is present at all 27 fixed points, the contribution $\mathcal{K}_2$ is present at $(0,0,n_3)$ and the contribution $\mathcal{K}_3$ is present at $(0,n_2,0)$. This leads to the tadpole equations 4.6. They have a unique solution. The results are displayed in tables 21 and 22.

In the cases without discrete torsion, the tadpole conditions for the $(0,2)$ sector are identical to eq. (4.6) if all $D5_i$-branes sit at the origin. Now, the remaining twisted sectors give additional conditions because, in general, the γ matrices are not traceless. The tadpole equations have many solutions, 1590 for the $(-1,-1,-1)$ model and 1295 for the $(+1,+1,-1)$ model. In general, the three sets of $D5_i$-branes have different gauge groups. However, if we impose an additional symmetry between the $D5_i$-branes, e.g. $\mathrm{Tr}(\gamma_{\bar{k},5_1}) = \mathrm{Tr}(\gamma_{\bar{k},5_2}) = \mathrm{Tr}(\gamma_{\bar{k},5_3})$ for odd $\bar{k}$ without fixed planes, then there is a unique solution. We display the 99 and 5_i5_i spectrum of these models in tables 23 and 24. We determined also the 95_i and 5_i5_j spectrum of these models and verified that the complete spectrum is free of non-abelian gauge anomalies.

4.6 $\mathbb{Z}_6 \times \mathbb{Z}_6$, $v = \frac{1}{6}(1,-1,0)$, $w = \frac{1}{6}(0,1,-1)$

This orientifold is symmetric under a permutation of the three sets of $D5_i$-branes. Thus, only four of the eight possible $\mathbb{Z}_6 \times \mathbb{Z}_6$ models are inequivalent. The $(+1,+1,+1)$ model has already been constructed in [6].

The only non-trivial tadpole conditions for the models with discrete torsion are (3.19), (3.23) and (3.21), corresponding to the untwisted sector, the the three

$\mathbb{Z}_2 \times \mathbb{Z}_6'$, $(\alpha_1, \alpha_2, \alpha_3) = (+1, +1, +1)$, $\epsilon = -1$	
open string spectrum	
sector	gauge group / matter fields
99, $5_i 5_i$ (susy)	$\mathrm{U}(6)_1 \times \mathrm{USp}(4)_2$ $3\,(\Box_1, \Box_2)$, $3\,\overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_1$
95_1, $5_2 5_3$ (susy)	$(\overline{\Box}_1, \overline{\Box}_1)$, $(\Box_1, \Box_2)$, $(\Box_2, \Box_1)$
95_2, $5_3 5_1$ (susy)	$(\Box_1, \Box_2)$, $(\overline{\Box}_1, \overline{\Box}_1)$, $(\Box_2, \Box_1)$
95_3, $5_1 5_2$ (susy)	$(\Box_1, \Box_2)$, $(\overline{\Box}_1, \overline{\Box}_1)$, $(\Box_2, \Box_1)$
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 3 chir.
order-two	15 chir.
remaining	12 lin.

Table 21: Spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_6'$ orientifold with discrete torsion and $(\mu_3, \mu_1) = (-1, -1)$. The notation is explained in the tables of section 4.1.

$\mathbb{Z}_2 \times \mathbb{Z}_6'$, $(\alpha_1, \alpha_2, \alpha_3) = (-1, +1, -1)$, $\epsilon = -1$	
open string spectrum	
sector	gauge group / matter fields
99, $5_2 5_2$ (susy)	$\mathrm{SO}(4)_1 \times \mathrm{U}(6)_2$ $3\,(\Box_1, \overline{\Box}_2)$, $2\,\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_2$, $\Box\!\Box_2$
$5_1 5_1$, $5_3 5_3$ (non-susy)	$\mathrm{USp}(4)_1 \times \mathrm{U}(6)_2$ $\mathcal{N} = 1$ mult.: $3\,(\Box_1, \overline{\Box}_2)$ spinors: $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_1$, adj_2, $2\,\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_2$, $\Box\!\Box_2$, scalars: $2\,\Box\!\Box_2$, $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_2$
95_1, $5_2 5_3$ (non-susy)	spinors: $(\Box_1, \Box_2)$, $(\Box_2, \Box_1)$, $(\overline{\Box}_2, \overline{\Box}_2)$ scalars: $(\Box_1, \Box_1)$, $(\Box_2, \overline{\Box}_2)$, $(\overline{\Box}_2, \Box_2)$
95_2, $5_3 5_1$ (susy)	$(\Box_1, \overline{\Box}_2)$, $(\Box_2, \Box_2)$, $(\overline{\Box}_2, \Box_1)$
95_3, $5_1 5_2$ (non-susy)	spinors: $(\Box_1, \Box_2)$, $(\overline{\Box}_2, \overline{\Box}_2)$, $(\Box_2, \Box_1)$ scalars: $(\Box_1, \Box_1)$, $(\Box_2, \overline{\Box}_2)$, $(\overline{\Box}_2, \Box_2)$
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 3 chir.
order-two	5 chir., 10 vec.
remaining	12 lin.

Table 22: Spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_6'$ orientifold with discrete torsion and $(\mu_3, \mu_1) = (+1, +1)$. The notation is explained in the tables of section 4.1.

sectors twisted by $\bar{k}_1 = (0, 2)$, $\bar{k}_2 = (2, 2)$, $\bar{k}_3 = (2, 0)$ and the $(2, 4)$ sector, respectively.[15] The untwisted tadpoles fix the rank of the gauge group to be 8 for each of

[15]In contrast to the notation used in the rest of this paper, the three twists $\bar{k}_i$ defined in this

$\mathbb{Z}_2\times\mathbb{Z}_6',\ (\alpha_1,\alpha_2,\alpha_3)=(-1,-1,-1),\ \epsilon=1$	
open string spectrum	
sector	gauge group / matter fields
99 (susy)	$\mathrm{SO}(2)_1\times\mathrm{U}(3)_2\times\mathrm{U}(3)_3\times\mathrm{SO}(2)_4$ $\times\,\mathrm{SO}(2)_5\times\mathrm{U}(3)_6\times\mathrm{U}(3)_7\times\mathrm{SO}(2)_8$ $(\Box_1,\overline{\Box}_2),\ (\Box_1,\overline{\Box}_6),\ (\Box_1,\Box_7),\ (\Box_2,\overline{\Box}_3),\ (\overline{\Box}_2,\Box_5),\ (\Box_2,\Box_6),$ $(\Box_2,\overline{\Box}_7),\ (\overline{\Box}_2,\Box_8),\ (\Box_3,\Box_4),\ (\Box_3,\Box_5),\ (\overline{\Box}_3,\overline{\Box}_7),\ (\Box_3,\Box_8),$ $(\Box_4,\overline{\Box}_6),\ (\Box_4,\Box_7),\ (\Box_5,\overline{\Box}_6),\ (\Box_6,\overline{\Box}_3),\ (\Box_6,\overline{\Box}_7),\ (\Box_7,\Box_8)$
5_i5_i (non-susy)	$\mathrm{U}(3)_1\times\mathrm{U}(2)_2\times\mathrm{U}(3)_3\times\mathrm{U}(3)_4\times\mathrm{U}(2)_5\times\mathrm{U}(3)_6$ $\mathcal{N}=1$ mult.: $(\Box_1,\overline{\Box}_2),\ (\overline{\Box}_1,\overline{\Box}_2),\ (\Box_1,\Box_3),\ (\overline{\Box}_1,\overline{\Box}_5),\ (\Box_1,\Box_6),\ (\Box_2,\overline{\Box}_3),$ $(\overline{\Box}_2,\overline{\Box}_4),\ (\Box_2,\Box_5),\ (\Box_3,\Box_4),\ (\Box_3,\overline{\Box}_4),\ (\overline{\Box}_3,\overline{\Box}_6),\ (\Box_4,\overline{\Box}_5),$ $(\overline{\Box}_4,\overline{\Box}_5),\ (\Box_4,\Box_6),\ (\Box_5,\overline{\Box}_6),\ (\Box_6,\overline{\Box}_1)$ spinors: $adj_1,\ \ldots,\ adj_6,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_2,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_3,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_5,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_6$ scalars: $\Box\!\Box_2,\ \overline{\Box\!\Box}_3,\ \Box\!\Box_5,\ \overline{\Box\!\Box}_6$
$95_i,\ 5_i5_j$	many bifundamentals
closed string spectrum	
sector	$\mathcal{N}=1$ multiplets
untw.	gravity, 1 lin., 3 chir.
order-two	18 lin.
remaining	15 lin.

Table 23: Spectrum of the $\mathbb{Z}_2\times\mathbb{Z}_6'$ orientifold without discrete torsion and $(\mu_3,\mu_1)=(+1,+1)$. In general, the tadpole equations have many solutions for the ranks of the gauge group factors. The most symmetric one is shown. The notation is explained in the tables of section 4.1.

the four sectors 99, 5_i5_i. The tadpole conditions of the $\bar{k}_i$ and $(2,4)$ sectors read

$$\begin{aligned}\mathrm{Tr}(\gamma_{\bar{k}_i,9}) &= \mathrm{Tr}(\gamma_{\bar{k}_i,5_i}) = -\,\mathrm{Tr}(\gamma_{\bar{k}_i,5_j}) = -\,\mathrm{Tr}(\gamma_{\bar{k}_i,5_k}) = -\alpha_i\,\epsilon^{-k_{i,1}k_{i,2}/4}\,8\,,\\ \mathrm{Tr}(\gamma_{(2,4),9}) &= -\,\mathrm{Tr}(\gamma_{(2,4),5_1}) = -\,\mathrm{Tr}(\gamma_{(2,4),5_2}) = \mathrm{Tr}(\gamma_{(2,4),5_3}) = \alpha_3\,4\,,\end{aligned}\tag{4.7}$$

where $k_{i,1}$, $k_{i,2}$ are the components of the two-vector $\bar{k}_i$ and (ijk) is a permutation of (123). We assumed that all D5$_i$-branes are at the origin. The sign $\epsilon^{-k_{i,1}k_{i,2}/4}$ is -1 for the $(2,2)$ sector in the models with discrete torsion and $+1$ else. In the case without discrete torsion, these equations are still valid. But in addition, one has to take into account the tadpole conditions for all the remaining twisted sectors. The tadpole equations can be solved with a computer algebra program. The complete spectrum of the two models with discrete torsion is displayed in tables 25 and 26.

paragraph are not of order two.

JHEP10(2000)049

$\mathbb{Z}_2 \times \mathbb{Z}'_6,\ (\alpha_1, \alpha_2, \alpha_3) = (+1, +1, -1),\ \epsilon = 1$	
open string spectrum	
sector	gauge group / matter fields
$99,\ 5_1 5_1,\ 5_2 5_2$ (susy)	$\mathrm{U}(3)_1 \times \mathrm{U}(2)_2 \times \mathrm{U}(3)_3 \times \mathrm{U}(3)_4 \times \mathrm{U}(2)_5 \times \mathrm{U}(3)_6$ $(\square_1, \square_2),\ (\square_1, \overline{\square}_2),\ (\overline{\square}_1, \overline{\square}_4),\ (\square_1, \square_5),\ (\square_2, \overline{\square}_3),\ (\overline{\square}_2, \overline{\square}_3),$ $(\square_2, \square_4),\ (\overline{\square}_2, \overline{\square}_6),\ (\square_3, \overline{\square}_4),\ (\overline{\square}_3, \overline{\square}_5),\ (\square_3, \square_6),\ (\square_4, \square_5),$ $(\square_4, \overline{\square}_5),\ (\square_5, \overline{\square}_6),\ (\overline{\square}_5, \overline{\square}_6),\ (\square_6, \overline{\square}_1),\ \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_1,\ \begin{smallmatrix}\square\\\square\end{smallmatrix}_3,\ \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_4,\ \begin{smallmatrix}\square\\\square\end{smallmatrix}_6$
$5_3 5_3$ (non-susy)	$\mathrm{USp}(2)_1 \times \mathrm{U}(3)_2 \times \mathrm{U}(3)_3 \times \mathrm{USp}(2)_4$ $\times\, \mathrm{USp}(2)_5 \times \mathrm{U}(3)_6 \times \mathrm{U}(3)_7 \times \mathrm{USp}(2)_8$ $\mathcal{N} = 1$ mult.: $(\square_1, \overline{\square}_2),\ (\square_1, \overline{\square}_6),\ (\square_1, \square_7),\ (\square_2, \overline{\square}_3),\ (\overline{\square}_2, \square_5),\ (\square_2, \square_6),$ $(\square_2, \overline{\square}_7),\ (\overline{\square}_2, \square_8),\ (\square_3, \square_4),\ (\square_3, \square_5),\ (\overline{\square}_3, \overline{\square}_7),\ (\square_3, \square_8),$ $(\square_4, \overline{\square}_6),\ (\square_4, \square_7),\ (\square_5, \overline{\square}_6),\ (\square_6, \overline{\square}_3),\ (\square_6, \overline{\square}_7),\ (\square_7, \square_8)$ spinors: $\begin{smallmatrix}\square\\\square\end{smallmatrix}_1,\ adj_2,\ adj_3,\ \begin{smallmatrix}\square\\\square\end{smallmatrix}_4,\ \begin{smallmatrix}\square\\\square\end{smallmatrix}_5,\ adj_6,\ adj_7,\ \begin{smallmatrix}\square\\\square\end{smallmatrix}_8$
$95_i,\ 5_i 5_j$	many bifundamentals
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 3 chir.
order-two	6 lin., 12 chir.
remaining	15 lin.

Table 24: Spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}'_6$ orientifold without discrete torsion and $(\mu_3, \mu_1) = (+1, -1)$. In general, the tadpole equations have many solutions for the ranks of the gauge group factors. The most symmetric one is shown. The notation is explained in the tables of section 4.1.

For the two models without discrete torsion, we restrict ourselves to give the 99 and $5_i 5_i$ spectrum in tables 27–30. We determined also the 95_i and $5_i 5_j$ spectrum of these models and verified that the complete spectrum is free of non-abelian gauge anomalies. In the case without discrete torsion, the tadpole equations have many solutions. However, if we impose an additional symmetry between the D5_i-branes, e.g. $\mathrm{Tr}(\gamma_{\bar{k}, 5_1}) = \mathrm{Tr}(\gamma_{\bar{k}, 5_2}) = \mathrm{Tr}(\gamma_{\bar{k}, 5_3})$ for odd $\bar{k}$ without fixed planes, then there is a unique solution. The ranks of the gauge group factors for this solution are as follows:

$$
\begin{aligned}
[n_1, n_2, \ldots, n_{20}] &= [2, 1, 1, 2, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 0, 2, 1, 1, 2]\,, \\
[m_1, m_2, \ldots, m_{18}] &= [1, 1, 0, 1, 1, 0, 2, 1, 1, 2, 1, 1, 1, 0, 1, 1, 0, 1]\,.
\end{aligned} \tag{4.8}
$$

5. Conclusions

We have constructed a large number of compact $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifold models, with N, M even. These models are labelled by three signs: the vector structure associated

$\mathbb{Z}_6 \times \mathbb{Z}_6,\ (\alpha_1, \alpha_2, \alpha_3) = (+1, +1, +1),\ \epsilon = -1$	
open string spectrum	
sector	gauge group / matter fields
$99,\ 5_i 5_i$ (susy)	$\mathrm{U}(2)_1 \times \mathrm{U}(2)_2 \times \mathrm{U}(2)_3 \times \mathrm{USp}(4)_4$ $(\square_1, \square_4),\ (\square_2, \overline{\square}_1),\ (\square_2, \square_3),\ (\overline{\square}_2, \square_4),\ (\square_3, \overline{\square}_1), (\overline{\square}_3, \square_4)$ $\overline{\boxminus}_1,\ \boxminus_2,\ \boxminus_3$
$95_1,\ 5_2 5_3$ (susy)	$(\square_1, \square_4),\ (\overline{\square}_1, \overline{\square}_1),\ (\square_2, \square_3),\ (\square_3, \square_2),\ (\square_4, \square_1)$
$95_2,\ 5_3 5_1$ (susy)	$(\overline{\square}_1, \square_3),\ (\overline{\square}_2, \square_4),\ (\square_2, \square_2),\ (\square_3, \overline{\square}_1),\ (\square_4, \overline{\square}_2)$
$95_3,\ 5_1 5_2$ (susy)	$(\overline{\square}_1, \square_2),\ (\square_2, \overline{\square}_1),\ (\overline{\square}_3, \square_4),\ (\square_3, \square_3),\ (\square_4, \overline{\square}_3)$
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 3 chir.
order-two	3 chir.
remaining	36 lin., 12 chir.

Table 25: Spectrum of the $\mathbb{Z}_6 \times \mathbb{Z}_6$ orientifold with discrete torsion and $(\mu_3, \mu_1) = (-1, -1)$. The notation is explained in the tables of section 4.1.

to the two generators of the orbifold group and the discrete torsion, i.e. eight models for each $\mathbb{Z}_N \times \mathbb{Z}_M$.

Some of these models can be supersymmetric, i.e. only D-branes need to be added in order to cancel the tadpoles. All supersymmetric models have discrete torsion and no vector structure in both generators of the orbifold group. These models are the $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_6$, $\mathbb{Z}_2 \times \mathbb{Z}'_6$, $\mathbb{Z}_6 \times \mathbb{Z}_6$.

We have extended the list of consistent orientifolds adding some models that need antibranes to cancel the RR tadpoles, along the lines of [20]. For most of these models there is at least one solution. However in one of the $\mathbb{Z}_2 \times \mathbb{Z}_4$ models and in one of the $\mathbb{Z}_4 \times \mathbb{Z}_4$ models without Wilson lines, there is an incompatibility between the twisted tadpoles related to different fixed points.

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A. How to obtain the Hodge numbers of orbifolds with and without discrete torsion

In this section, we explain how to compute the Hodge numbers of compact orbifold

JHEP10(2000)049

$\mathbb{Z}_6\times\mathbb{Z}_6,\ (\alpha_1,\alpha_2,\alpha_3)=(-1,+1,-1),\ \epsilon=-1$	
open string spectrum	
sector	gauge group / matter fields
$99,\ 5_25_2$ (susy)	$\mathrm{SO}(4)_1\times\mathrm{U}(2)_2\times\mathrm{U}(2)_3\times\mathrm{U}(2)_4$ $(\square_1,\square_2),\ (\square_1,\overline{\square}_3),\ (\square_1,\square_4),\ (\overline{\square}_2,\overline{\square}_4),(\square_3,\overline{\square}_2),(\square_3,\overline{\square}_4)$ $\overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}_2,\ \begin{smallmatrix}\square\\ \square\end{smallmatrix}_3,\ \overline{\square\!\square}_4$
$5_15_1,\ 5_35_3$ (non-susy)	$\mathrm{USp}(4)_1\times\mathrm{U}(2)_2\times\mathrm{U}(2)_3\times\mathrm{U}(2)_4$ $\mathcal{N}=1$ mult.: identical to bifundamentals of 99 spinors: $\begin{smallmatrix}\square\\ \square\end{smallmatrix}_1,\ adj_2,\ adj_3,\ adj_4,\ \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}_2,\ \begin{smallmatrix}\square\\ \square\end{smallmatrix}_3,\ \overline{\square\!\square}_4$ scalars: $\overline{\square\!\square}_2,\ \square\!\square_3,\ \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}_4$
$95_1,\ 5_25_3$ (non-susy)	spinors: $(\square_1,\square_3),\ (\square_2,\square_4),\ (\overline{\square}_3,\overline{\square}_3),\ (\square_3,\square_1),\ (\square_4,\square_2)$ scalars: $(\square_2,\square_3),\ (\overline{\square}_2,\square_4),\ (\overline{\square}_3,\overline{\square}_4),\ (\square_3,\square_2),\ (\overline{\square}_4,\overline{\square}_3),\ (\square_4,\overline{\square}_2)$
$95_2,\ 5_35_1$ (susy)	$(\square_1,\square_4),\ (\overline{\square}_2,\square_3),\ (\square_3,\overline{\square}_2),\ (\overline{\square}_4,\overline{\square}_4),\ (\square_4,\square_1)$
$95_3,\ 5_15_2$ (non-susy)	spinors: $(\square_1,\overline{\square}_2),\ (\square_2,\square_2),\ (\overline{\square}_2,\square_1),\ (\overline{\square}_3,\square_4),\ (\square_4,\overline{\square}_3)$ scalars: $(\overline{\square}_2,\square_4),\ (\square_2,\square_3),\ (\square_3,\square_2),\ (\overline{\square}_3,\overline{\square}_4),\ (\square_4,\overline{\square}_2),\ (\overline{\square}_4,\overline{\square}_3)$
closed string spectrum	
sector	$\mathcal{N}=1$ multiplets
untw.	gravity, 1 lin., 3 chir.
order-two	1 chir., 2 vec.
remaining	36 lin., 12 chir.

Table 26: Spectrum of the $\mathbb{Z}_6\times\mathbb{Z}_6$ orientifold with discrete torsion and $(\mu_3,\mu_1)=(+1,+1)$. The notation is explained in the tables of section 4.1.

spaces $\mathbb{T}^6/\Gamma$, where the six-torus $\mathbb{T}^6$ is of the form $\mathbb{C}^3/\Lambda$ and Γ is an abelian finite group. The Hodge number $h^{p,q}$ is defined as the number of independent harmonic (p,q)-forms that can be defined on this space.

An element $g\in\Gamma$ acts on the three complex coordinates of $\mathbb{C}^3/\Lambda$ as

$$g:\ (z_1,z_2,z_3)\ \longrightarrow\left(e^{2\pi i v_g^{(1)}}z_1,e^{2\pi i v_g^{(2)}}z_2,e^{2\pi i v_g^{(3)}}z_3\right),\quad \text{with } 0\le v_g^{(i)}<1\,. \tag{A.1}$$

We will be interested in the separate contributions of each twisted sector to $h^{p,q}$:

$$h^{p,q}=h^{p,q}_{\text{untw}}+\sum_{g\in\Gamma\backslash\{e\}}h^{p,q}_g\,. \tag{A.2}$$

The contribution of the untwisted sector is just the number of Γ-invariant harmonic (p,q)-forms that can be defined on $\mathbb{T}^6$. The forms on $\mathbb{T}^6$ are generated by

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & dz_i & & d\bar{z}_i & & \\ & dz_idz_j & & dz_id\bar{z}_j & & d\bar{z}_id\bar{z}_j & \\ dz_idz_jdz_k & & dz_idz_jd\bar{z}_k & & dz_id\bar{z}_jd\bar{z}_k & & d\bar{z}_id\bar{z}_jd\bar{z}_k \\ & & & \vdots & & & \end{array}$$

JHEP10(2000)049

$\mathbb{Z}_6 \times \mathbb{Z}_6,\ (\alpha_1,\alpha_2,\alpha_3)=(-1,-1,-1),\ \epsilon=1$	
open string spectrum	
sector	gauge group / matter fields
99	$\mathrm{SO}(n_1)\times U(n_2)\times U(n_3)\times \mathrm{SO}(n_4)\times U(n_5)$
	$\times U(n_6)\times U(n_7)\times U(n_8)\times U(n_9)\times U(n_{10})$
	$\times U(n_{11})\times U(n_{12})\times U(n_{13})\times U(n_{14})\times U(n_{15})$
	$\times U(n_{15})\times \mathrm{SO}(n_{17})\times U(n_{18})\times U(n_{19})\times \mathrm{SO}(n_{20})$
(susy)	$(\Box_1,\Box_2),\ (\Box_1,\overline{\Box}_5),\ (\Box_1,\Box_{10}),\ (\Box_2,\overline{\Box}_6),\ (\Box_2,\Box_9),\ (\overline{\Box}_2,\overline{\Box}_{10}),$
	$(\Box_3,\overline{\Box}_2),\ (\overline{\Box}_3,\Box_4),\ (\Box_3,\overline{\Box}_7),\ (\Box_3,\Box_8),\ (\overline{\Box}_3,\overline{\Box}_9),\ (\Box_4,\Box_7),$
	$(\Box_4,\overline{\Box}_8),\ (\Box_5,\overline{\Box}_2),\ (\Box_5,\overline{\Box}_{10}),\ (\Box_5,\overline{\Box}_{11}),\ (\Box_6,\overline{\Box}_3),\ (\Box_6,\overline{\Box}_5),$
	$(\Box_6,\overline{\Box}_{12}),\ (\Box_7,\overline{\Box}_6),\ (\Box_7,\overline{\Box}_{13}),\ (\Box_8,\overline{\Box}_7),\ (\Box_8,\overline{\Box}_{14}),\ (\Box_9,\overline{\Box}_8),$
	$(\Box_9,\overline{\Box}_{15}),\ (\Box_{10},\overline{\Box}_9),\ (\Box_{10},\overline{\Box}_{15}),\ (\Box_{11},\overline{\Box}_6),\ (\Box_{11},\overline{\Box}_{15}),\ (\Box_{11},\Box_{17}),$
	$(\overline{\Box}_{11},\overline{\Box}_{18}),\ (\Box_{12},\overline{\Box}_7),\ (\Box_{12},\overline{\Box}_{11}),\ (\overline{\Box}_{12},\Box_{17}),\ (\Box_{12},\overline{\Box}_{18}),\ (\Box_{13},\overline{\Box}_8),$
	$(\Box_{13},\overline{\Box}_{12}),\ (\Box_{13},\overline{\Box}_{19}),\ (\Box_{14},\overline{\Box}_9),\ (\Box_{14},\overline{\Box}_{13}),\ (\Box_{14},\Box_{20}),\ (\Box_{15},\overline{\Box}_{10}),$
	$(\Box_{15},\overline{\Box}_{14}),\ (\Box_{15},\Box_{19}),\ (\overline{\Box}_{15},\Box_{20}),\ (\Box_{15},\overline{\Box}_5),\ (\Box_{15},\overline{\Box}_{15}),\ (\Box_{15},\Box_{18}),$
	$(\overline{\Box}_{15},\overline{\Box}_{19}),\ (\Box_{17},\Box_{18}),\ (\Box_{18},\overline{\Box}_{13}),\ (\Box_{19},\overline{\Box}_{14}),\ (\Box_{19},\overline{\Box}_{18}),\ (\overline{\Box}_{19},\Box_{20})$

Table 27: Spectrum of the $\mathbb{Z}_6 \times \mathbb{Z}_6$ orientifold without discrete torsion and $(\mu_3,\mu_1) = (+1,+1)$. The numbers n_i are determined by solving the tadpole equations. There are many solutions. The most symmetric one is given in the main text, eq. (4.8). In general, several of the n_i vanish because the total rank of the 99 gauge group must be 16.

The contributions of the twisted sectors are due to the singularities of the orbifold which arise because $\mathbb{T}^6$ has fixed points or fixed planes under the action of the elements of Γ. We split the twisted sectors into two sets:

(a) Sectors twisted by g, where g is a group element that only has fixed points, but no fixed planes. This sector only contributes to $h^{1,1}$ or $h^{2,2}$. If $\sum_{i=1}^3 v_g^{(i)} = 1$, then [35, 22]

$$h_g^{1,1} = \frac{1}{|\Gamma|}\sum_{h\in\Gamma}\beta_{g,h}\,\chi(g,h)\,, \qquad h_g^{2,2}=0\,, \tag{A.3}$$

where $|\Gamma|$ is the number of elements of the discrete group, $\beta_{g,h}$ is the discrete torsion phase and $\chi(g,h)$ is the Euler characteristic of the subspace (i.e. the set of fixed points) left simultaneously fixed by g and h. In our case, $\chi(g,h)$ is the number of points that are simultaneously fixed by g and h. If $\sum_{i=1}^3 v_g^{(i)} = 2$, then this sector contributes the same value, but to $h_g^{2,2}$, and $h_g^{1,1}=0$.

The sector twisted by g^{-1} gives the same contribution to $h^{2,2}$ as the g-twisted sector to $h^{1,1}$ and vice versa:

$$h_{g^{-1}}^{2,2} = h_g^{1,1}\,, \qquad h_{g^{-1}}^{1,1} = h_g^{2,2}\,. \tag{A.4}$$

$\mathbb{Z}_6 \times \mathbb{Z}_6,\ (\alpha_1, \alpha_2, \alpha_3) = (-1, -1, -1),\ \epsilon = 1$	
open string spectrum	
sector	gauge group / matter fields
$5_i 5_i$ (non-susy)	$U(m_1) \times U(m_2) \times U(m_3) \times U(m_4) \times U(m_5) \times U(m_6)$ $\times U(m_7) \times U(m_8) \times U(m_9) \times U(m_{10}) \times U(m_{11}) \times U(m_{12})$ $\times U(m_{13}) \times U(m_{14}) \times U(m_{15}) \times U(m_{16}) \times U(m_{17}) \times U(m_{18})$ $\mathcal{N} = 1$ mult.: $(\Box_1, \Box_6)$, $(\Box_1, \overline{\Box}_6)$, $(\Box_1, \overline{\Box}_7)$, $(\Box_2, \overline{\Box}_1)$, $(\Box_2, \Box_5)$, $(\overline{\Box}_2, \overline{\Box}_6)$, $(\Box_2, \overline{\Box}_8)$, $(\Box_3, \overline{\Box}_2)$, $(\Box_3, \Box_4)$, $(\overline{\Box}_3, \overline{\Box}_5)$, $(\Box_3, \overline{\Box}_9)$, $(\Box_4, \overline{\Box}_3)$, $(\Box_4, \overline{\Box}_{10})$, $(\Box_5, \overline{\Box}_4)$, $(\Box_5, \overline{\Box}_{11})$, $(\Box_6, \overline{\Box}_5)$, $(\Box_6, \overline{\Box}_{12})$, $(\Box_7, \overline{\Box}_2)$, $(\Box_7, \overline{\Box}_{12})$, $(\Box_7, \overline{\Box}_{13})$, $(\Box_8, \overline{\Box}_3)$, $(\Box_8, \overline{\Box}_7)$, $(\Box_8, \overline{\Box}_{14})$, $(\Box_9, \overline{\Box}_4)$, $(\Box_9, \overline{\Box}_8)$, $(\Box_9, \overline{\Box}_{15})$, $(\Box_{10}, \overline{\Box}_5)$, $(\Box_{10}, \overline{\Box}_9)$, $(\Box_{10}, \overline{\Box}_{16})$, $(\Box_{11}, \overline{\Box}_6)$, $(\Box_{11}, \overline{\Box}_{10})$, $(\Box_{11}, \overline{\Box}_{17})$, $(\Box_{12}, \overline{\Box}_1)$, $(\Box_{12}, \overline{\Box}_{11})$, $(\Box_{12}, \overline{\Box}_{18})$, $(\Box_{13}, \overline{\Box}_8)$, $(\overline{\Box}_{13}, \overline{\Box}_{14})$, $(\Box_{13}, \overline{\Box}_{18})$, $(\Box_{14}, \overline{\Box}_9)$, $(\Box_{14}, \overline{\Box}_{13})$, $(\Box_{14}, \Box_{18})$, $(\Box_{15}, \overline{\Box}_{10})$, $(\Box_{15}, \overline{\Box}_{14})$, $(\Box_{15}, \Box_{17})$, $(\overline{\Box}_{15}, \overline{\Box}_{18})$, $(\Box_{16}, \overline{\Box}_{11})$, $(\Box_{16}, \overline{\Box}_{15})$, $(\overline{\Box}_{16}, \overline{\Box}_{17})$, $(\Box_{17}, \overline{\Box}_{12})$, $(\Box_{17}, \overline{\Box}_{16})$, $(\Box_{18}, \overline{\Box}_7)$, $(\Box_{18}, \overline{\Box}_{17})$ spinors: adj_1, ..., adj_{18}, $\overline{\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}}_1$, $\overline{\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}}_4$, $\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}_{13}$, $\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}_{16}$ scalars: $\overline{\Box\Box}_1$, $\overline{\Box\Box}_4$, $\Box\Box_{13}$, $\Box\Box_{16}$
$95_i,\ 5_i 5_j$	many bifundamentals
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 3 chir.
order-two	12 lin.
remaining	54 lin., 15 chir.

Table 28: Spectrum of the $\mathbb{Z}_6 \times \mathbb{Z}_6$ orientifold without discrete torsion and $(\mu_3, \mu_1) = (+1, +1)$. (continued)

(b) Sectors twisted by g, where g is a group element that has fixed planes. This sector gives a contribution to $h^{1,1}$, $h^{1,2}$, $h^{2,1}$ and $h^{2,2}$ of the following form [35, 22]:

$$
\begin{aligned}
h_g^{2,2} = h_g^{1,1} &= \frac{1}{|\Gamma|} \sum_{h \in \Gamma} \beta_{g,h}\, \widetilde{\chi}(g,h), \\
h_g^{1,2} &= \frac{1}{|\Gamma|} \sum_{h \in \Gamma} \beta_{g,h}\, \widetilde{\chi}(g,h)\, e^{2\pi i v_h(g)}, \\
h_g^{2,1} &= \frac{1}{|\Gamma|} \sum_{h \in \Gamma} \beta_{g,h}\, \widetilde{\chi}(g,h)\, e^{-2\pi i v_h(g)}\,,
\end{aligned} \tag{A.5}
$$

where $v_h(g) = v_h^{(i)}$ if the i-th plane is fixed by g, and $\widetilde{\chi}(g,h)$ is the Euler characteristic of the subspace that is simultaneously fixed under g and h and that is contained in the two internal complex planes which are not fixed under g (see [35] for a proper definition). The phase that appears in the formulae for $h_g^{1,2}$ and $h_g^{2,1}$ corresponds to

$\mathbb{Z}_6 \times \mathbb{Z}_6,\ (\alpha_1, \alpha_2, \alpha_3) = (+1, +1, -1),\ \epsilon = 1$	
open string spectrum	
sector	gauge group / matter fields
$99,\ 5_1 5_1,\ 5_2 5_2$	$U(m_1) \times U(m_2) \times U(m_3) \times U(m_4) \times U(m_5) \times U(m_6)$ $\times U(m_7) \times U(m_8) \times U(m_9) \times U(m_{10}) \times U(m_{11}) \times U(m_{12})$ $\times U(m_{13}) \times U(m_{14}) \times U(m_{15}) \times U(m_{16}) \times U(m_{17}) \times U(m_{18})$
(susy)	$(\Box_1, \Box_6),\ (\Box_1, \overline{\Box}_6),\ (\Box_1, \overline{\Box}_7),\ (\Box_2, \overline{\Box}_1),\ (\Box_2, \Box_5),\ (\overline{\Box}_2, \overline{\Box}_6),$ $(\Box_2, \overline{\Box}_8),\ (\Box_3, \overline{\Box}_2),\ (\Box_3, \Box_4),\ (\overline{\Box}_3, \overline{\Box}_5),\ (\Box_3, \overline{\Box}_9),\ (\Box_4, \overline{\Box}_3),$ $(\Box_4, \overline{\Box}_{10}),\ (\Box_5, \overline{\Box}_4),\ (\Box_5, \overline{\Box}_{11}),\ (\Box_6, \overline{\Box}_5),\ (\Box_6, \overline{\Box}_{12}),\ (\Box_7, \overline{\Box}_2),$ $(\Box_7, \overline{\Box}_{12}),\ (\Box_7, \overline{\Box}_{13}),\ (\Box_8, \overline{\Box}_3),\ (\Box_8, \overline{\Box}_7),\ (\Box_8, \overline{\Box}_{14}),\ (\Box_9, \overline{\Box}_4),$ $(\Box_9, \overline{\Box}_8),\ (\Box_9, \overline{\Box}_{15}),\ (\Box_{10}, \overline{\Box}_5),\ (\Box_{10}, \overline{\Box}_9),\ (\Box_{10}, \overline{\Box}_{16}),\ (\Box_{11}, \overline{\Box}_6),$ $(\Box_{11}, \overline{\Box}_{10}),\ (\Box_{11}, \overline{\Box}_{17}),\ (\Box_{12}, \overline{\Box}_1),\ (\Box_{12}, \overline{\Box}_{11}),\ (\Box_{12}, \overline{\Box}_{18}),\ (\Box_{13}, \overline{\Box}_8),$ $(\overline{\Box}_{13}, \overline{\Box}_{14}),\ (\Box_{13}, \overline{\Box}_{18}),\ (\Box_{14}, \overline{\Box}_9),\ (\Box_{14}, \overline{\Box}_{13}),\ (\Box_{14}, \Box_{18}),\ (\Box_{15}, \overline{\Box}_{10}),$ $(\Box_{15}, \overline{\Box}_{14}),\ (\Box_{15}, \Box_{17}),\ (\overline{\Box}_{15}, \overline{\Box}_{18}),\ (\Box_{16}, \overline{\Box}_{11}),\ (\Box_{16}, \overline{\Box}_{15}),\ (\overline{\Box}_{16}, \overline{\Box}_{17}),$ $(\Box_{17}, \overline{\Box}_{12}),\ (\Box_{17}, \overline{\Box}_{16}),\ (\Box_{18}, \overline{\Box}_7),\ (\Box_{18}, \overline{\Box}_{17}),\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_1,\ \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_4,\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_{13},\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_{16}$

Table 29: Spectrum of the $\mathbb{Z}_6 \times \mathbb{Z}_6$ orientifold without discrete torsion and $(\mu_3, \mu_1) = (+1, -1)$. The numbers m_i are determined by solving the tadpole equations. There are many solutions. In general, the three sectors 99, $5_1 5_1$ and $5_2 5_2$ have different gauge groups. The most symmetric solution is given in the main text, eq. (4.8). In general, several of the m_i vanish because the total rank of the each of the 99 and $5_i 5_i$ gauge groups must be 16.

the phase acquired by the forms dz_i and $d\bar{z}_i$ (defined on the plane that is fixed by g) under a twist by h. Note that in the case with discrete torsion the contributions to the Hodge numbers $h^{1,2}$ and $h^{2,1}$ can be different. But one always has: $h^{1,2}_g + h^{1,2}_{g^{-1}} = h^{2,1}_g + h^{2,1}_{g^{-1}}$.

To illustrate this method, let us analyse the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold. From the untwisted sector, one has the universal contribution (i.e. present in any orbifold): $h^{0,0} = h^{3,3} = h^{3,0} = h^{0,3} = 1$. The contribution of each sector to $h^{1,1}$, $h^{2,2}$, $h^{1,2}$ and $h^{2,1}$ in the cases with and without discrete torsion is shown in table 31.

B. Closed string spectrum from shifts

In this appendix we shall confirm the result on the closed string spectrum of type-IIB and type-I orbifolds that was obtained by analysing the cohomology as described in section 3.1. Let us consider the RR part of the spectrum.

We divide the twisted sectors into two sets:

(a) Sectors without fixed tori:

Let $v = (0, v_1, v_2, v_3)$ be the shift vector corresponding to this sector, with $0 \leq$

$\mathbb{Z}_6 \times \mathbb{Z}_6,\ (\alpha_1, \alpha_2, \alpha_3) = (+1, +1, -1),\ \epsilon = 1$	
open string spectrum	
sector	gauge group / matter fields
$5_3 5_3$ (non-susy)	$\mathrm{USp}(n_1) \times U(n_2) \times U(n_3) \times \mathrm{USp}(n_4) \times U(n_5)$ $\times U(n_6) \times U(n_7) \times U(n_8) \times U(n_9) \times U(n_{10})$ $\times U(n_{11}) \times U(n_{12}) \times U(n_{13}) \times U(n_{14}) \times U(n_{15})$ $\times U(n_{16}) \times \mathrm{USp}(n_{17}) \times U(n_{18}) \times U(n_{19}) \times \mathrm{USp}(n_{20})$ $\mathcal{N} = 1$ mult.: $(\Box_1, \Box_2)$, $(\Box_1, \overline{\Box}_5)$, $(\Box_1, \Box_{10})$, $(\Box_2, \overline{\Box}_6)$, $(\Box_2, \Box_9)$, $(\overline{\Box}_2, \overline{\Box}_{10})$, $(\Box_3, \overline{\Box}_2)$, $(\overline{\Box}_3, \Box_4)$, $(\Box_3, \overline{\Box}_7)$, $(\Box_3, \Box_8)$, $(\overline{\Box}_3, \overline{\Box}_9)$, $(\Box_4, \Box_7)$, $(\Box_4, \overline{\Box}_8)$, $(\Box_5, \overline{\Box}_2)$, $(\Box_5, \Box_{10})$, $(\Box_5, \overline{\Box}_{11})$, $(\Box_6, \overline{\Box}_3)$, $(\Box_6, \overline{\Box}_5)$, $(\Box_6, \overline{\Box}_{12})$, $(\Box_7, \overline{\Box}_6)$, $(\Box_7, \Box_{13})$, $(\Box_8, \overline{\Box}_7)$, $(\Box_8, \overline{\Box}_{14})$, $(\Box_9, \overline{\Box}_8)$, $(\Box_9, \overline{\Box}_{15})$, $(\Box_{10}, \overline{\Box}_9)$, $(\Box_{10}, \overline{\Box}_{16})$, $(\Box_{11}, \overline{\Box}_6)$, $(\Box_{11}, \overline{\Box}_{16})$, $(\Box_{11}, \Box_{17})$, $(\overline{\Box}_{11}, \overline{\Box}_{18})$, $(\Box_{12}, \overline{\Box}_7)$, $(\Box_{12}, \overline{\Box}_{11})$, $(\overline{\Box}_{12}, \Box_{17})$, $(\Box_{12}, \overline{\Box}_{18})$, $(\Box_{13}, \overline{\Box}_8)$, $(\Box_{13}, \overline{\Box}_{12})$, $(\Box_{13}, \overline{\Box}_{19})$, $(\Box_{14}, \overline{\Box}_9)$, $(\Box_{14}, \overline{\Box}_{13})$, $(\Box_{14}, \Box_{20})$, $(\Box_{15}, \overline{\Box}_{10})$, $(\Box_{15}, \overline{\Box}_{14})$, $(\Box_{15}, \Box_{19})$, $(\overline{\Box}_{15}, \Box_{20})$, $(\Box_{16}, \overline{\Box}_5)$, $(\Box_{16}, \overline{\Box}_{15})$, $(\Box_{16}, \Box_{18})$, $(\overline{\Box}_{16}, \overline{\Box}_{19})$, $(\Box_{17}, \Box_{18})$, $(\Box_{18}, \overline{\Box}_{13})$, $(\Box_{19}, \overline{\Box}_{14})$, $(\Box_{19}, \overline{\Box}_{18})$, $(\overline{\Box}_{19}, \Box_{20})$ spinors: $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_1$, adj_2, adj_3, $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_4$, adj_5, $\ldots$, adj_{16}, $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_{17}$, adj_{18}, adj_{19}, $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_{20}$
$95_i,\ 5_i 5_j$	many bifundamentals
closed string spectrum	
sector	$\mathcal{N} = 1$ multiplets
untw.	gravity, 1 lin., 3 chir.
order-two	4 lin., 8 chir.
remaining	54 lin., 15 chir.

Table 30: Spectrum of the $\mathbb{Z}_6 \times \mathbb{Z}_6$ orientifold without discrete torsion and $(\mu_3, \mu_1) = (+1, -1)$. (continued)

$v_i < 1$. The vacuum energy corresponding to this sector is

$$E_0 = \frac{1}{2} \sum_i v_i (1 - v_i)\,. \tag{B.1}$$

The RR states are characterised by SO(8) weight vectors of the form $r = (\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)$, with an odd number of minus signs because of the GSO projection. The mass of a state characterised by the weight vector r_v is given by

$$M^2 = \frac{(r_v + v)^2}{2} - \frac{1}{2} + E_0\,. \tag{B.2}$$

There is only one massless state (having imposed the GSO projection): $r_v = (1/2, -1/2, -1/2, -1/2)$.

In type-IIB string theory, there is only one RR state in this sector $r_{v,L} \otimes r_{v,R}$, with helicity $\chi = r^0_{v,L} - r^0_{v,R} = 0$. This is a scalar or its dual 2-form. From the sector

$\mathbb{Z}_2 \times \mathbb{Z}_4$		$\epsilon = 1$				$\epsilon = -1$			
g	fixed plane	$h_g^{1,1}$	$h_g^{2,2}$	$h_g^{1,2}$	$h_g^{2,1}$	$h_g^{1,1}$	$h_g^{2,2}$	$h_g^{1,2}$	$h_g^{2,1}$
(0,0)	all	3	3	1	1	3	3	1	1
(0,1)	1st	4	4	-	-	-	-	4	4
(0,2)	1st	10	10	-	-	10	10	-	-
(0,3)	1st	4	4	-	-	-	-	4	4
(1,0)	3rd	12	12	-	-	4	4	-	-
(1,1)	-	-	16	-	-	-	-	-	-
(1,2)	2nd	12	12	-	-	4	4	-	-
(1,3)	-	16	-	-	-	-	-	-	-
total		61	61	1	1	21	21	9	9

Table 31: Hodge numbers of the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold with and without discrete torsion.

with shift $-v$, one obtains another solution: $r_{-v} = (-1/2, +1/2, +1/2, +1/2)$. This gives another scalar (or a 2-form). The degeneracy of these states can be obtained from the partition function, as explained in [35, 21]. For the total number of states, one finds:

$$n_{\rm scal}(v) = \frac{2}{|\Gamma|} \sum_{h \in \Gamma} \beta_{v,h}\, \chi(v,h)\,, \tag{B.3}$$

where it is understood that we only sum over half the shifts to get the total number of scalars (because we combined v and $-v$). This corresponds to the $h_v^{1,1}$ scalars and 2-forms (see the formula (A.3) for the Hodge numbers $h_g^{1,1}$) coming from the reduction of the RR 2-form and 4-form. Adding the scalars from the NSNS sector and the corresponding fermions, we obtain $h_v^{1,1}$ $\mathcal{N} = 2$ tensor multiplets, as predicted by the cohomology computation.

In type-I string theory, there is an Ω projection that exchanges left-movers and right-movers. It acts together with a J operation that exchanges v and $-v$. As we are in the RR sector, we must take the antisymmetric combinations: $r_{v,L} \otimes r_{v,R} - r_{-v,L} \otimes r_{-v,R}$. Only a scalar (or a 2-form) remains. The degeneracy is again $h_v^{1,1}$, i.e. half the value of (B.3), and coincides with the result from the cohomology computation. Together with the NSNS and fermionic states, one finds $h_v^{1,1}$ $\mathcal{N} = 1$ linear multiplets.
(b) Sectors with a fixed torus:

Let $v = (0, 0, v_2, v_3)$ be the shift corresponding to this sector, with $0 \leq v_i < 1$. The first torus is fixed.

There are two massless states: $r_+ = (1/2, -1/2, 1/2, 1/2)$ and $r_- = (-1/2, 1/2, 1/2, 1/2)$. (The GSO projection has been imposed.)

In type-IIB string theory, there are four RR states in this sector:

- $r_{+,L} \otimes r_{+,R}$ with helicity: $\chi = r_L^0 - r_R^0 = 0\,,$
- $r_{-,L} \otimes r_{-,R}$ with helicity: $\chi = r_L^0 - r_R^0 = 0\,,$

- $r_{+,L} \otimes r_{-,R}$ with helicity: $\chi = r_L^0 - r_R^0 = 1$,
- $r_{-,L} \otimes r_{+,R}$ with helicity: $\chi = r_L^0 - r_R^0 = -1$.

This is a pair of scalars (or 2-forms) and a vector. The number of scalars is:

$$n_{\mathrm{scal}}(v) = \frac{2}{|\Gamma|} \sum_{h \in \Gamma} \beta_{v,h}\, \widetilde{\chi}(v,h)\,. \tag{B.4}$$

This corresponds to the $h_v^{1,1}$ scalars and 2-forms (see the formula (A.5) for the Hodge numbers $h_g^{1,1}$) coming from the reduction of the RR 2-forms and 4-forms, as predicted by the cohomology computation. The number of vectors is:

$$n_{\mathrm{vec}}(v) = \frac{1}{|\Gamma|} \sum_{h \in \Gamma} \beta_{v,h}\, \widetilde{\chi}(v,h)\, e^{2\pi i v_h^{(1)}}\,, \tag{B.5}$$

where $v_h^{(1)}$ is the first component of the shift vector corresponding to the twist h (in general one has to take $v_h^{(i)}$ if the i-th torus is fixed). These are $h_v^{1,2}$ vectors from the reduction of the RR 4-form.

In type-I string theory, one must distinguish between the sectors of order different from two and of order two. The latter are mapped onto themselves under J. For sectors that are not of order two, the sectors v and $-v$ are combined under the J operation, leading to four linear combinations of type-IIB RR states:

- $r^v_{+,L} \otimes r^v_{+,R} \;-\; r^{-v}_{-,L} \otimes r^{-v}_{-,R}$ with helicity: $\chi = r_L^0 - r_R^0 = 0$,
- $r^v_{-,L} \otimes r^v_{-,R} \;-\; r^{-v}_{+,L} \otimes r^{-v}_{+,R}$ with helicity: $\chi = r_L^0 - r_R^0 = 0$,
- $r^v_{+,L} \otimes r^v_{-,R} \;-\; r^{-v}_{+,L} \otimes r^{-v}_{-,R}$ with helicity: $\chi = r_L^0 - r_R^0 = 1$,
- $r^v_{-,L} \otimes r^v_{+,R} \;-\; r^{-v}_{-,L} \otimes r^{-v}_{+,R}$ with helicity: $\chi = r_L^0 - r_R^0 = -1$.

Note that, because of the J operation, the pairing of states we have performed above is only possible if $h_v^{1,2} = h_v^{2,1}$. From (B.5), one finds that this is equivalent to the statement that the discrete torsion $\beta_{v,h}$ only takes real values, i.e. $\beta_{v,h} = \pm 1$.

If the sector is of order two, then v is identical to $-v$. Of the four states that survive in the general case only one remains:

- $r^v_{+,L} \otimes r^v_{+,R} \;-\; r^{-v}_{-,L} \otimes r^{-v}_{-,R}$ with helicity: $\chi = r_L^0 - r_R^0 = 0$.

This corresponds to the $h_v^{1,1}$ scalars from the reduction of the RR 2-form.

We see that the results from the shift formalism coincide with those obtained from the cohomology computation.

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